From Petrov-Einstein to Navier-Stokes

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Abstract

The fluid/gravity correspondence relates solutions of the incompressible Navier-Stokes equation to metrics which solve the Einstein equations. We propose propose two possible approaches to establish this correspondence: perturbative expansion for shear modes and large mean curvature expansion for algebraically special metrics.

We show by explicit construction that for every solution of the incompressible Navier-Stokes equation in p+1 dimensions, there is an associated dual solution of the vacuum Einstein equations in p+2 dimensions. The dual geometry has an intrinsically flat time-like boundary segment whose extrinsic curvature is given by the stress tensor of the Navier-Stokes fluid. We consider a near-horizon limit in which hypersurface becomes highly accelerated. The near-horizon expansion in gravity is shown to be mathematically equivalent to the hydrodynamic expansion in fluid dynamics, and the Einstein equation reduces to the incompressible Navier-Stokes equation.

It is shown that imposing a Petrov type I condition on the hypersurface geometry reduces the degrees of freedom in the extrinsic curvature to those of a fluid. Moreover, expanding around a limit in which the mean curvature of the embedding diverges, the leading-order Einstein constraint equations on hypersurface are shown to reduce to the non-linear incompressible Navier-Stokes equation for a fluid moving in hypersurface.

We extend the fluid/gravity correspondence to include the magnetohydrodynamics/gravity correspondence, which translates solutions of the equations of magnetohydrodynamics (describing charged fluids) into geometries that satisfy the Einstein-Maxwell equations. We present an explicit example of this new correspondence in the context of flat Minkowski space. We show that a perturbative deformation of the Rindler wedge satisfies the Einstein-Maxwell equations provided that the parameters appearing in the expansion, which we interpret as fluid fields, satisfy the magnetohydrodynamics equations.

Contents

Citations to Previously Published Work

Most of the Chapters and Appendices of this thesis has been appeared in print elsewhere. Below the list, by Chapter and Appendix number, of previously published work.

- Chapter 3: "Wilsonian Approach to Fluid/Gravity Duality," I. Bredberg, C. Keeler, V. Lysov and A. Strominger, JHEP **1103**, 141 (2011) `arXiv:1006.1902 [hep-th]`.
- Chapter 4: "From Navier-Stokes To Einstein", I. Bredberg, C. Keeler, V. Lysov and A. Strominger, JHEP **1207**, 146 (2012) `arXiv:1101.2451 [hep-th]`.
- Chapter 5: "From Petrov-Einstein to Navier-Stokes," V. Lysov and A. Strominger, `arXiv:1104.5502 [hep-th]`.
- Chapter 6: "On the Magnetohydrodynamics/Gravity Correspondence," V. Lysov, `arXiv:1310.4181 [hep-th]`.

Electronic preprints (shown in `typewriter font`) are available on the Internet at the following URL:

`http://arXiv.org`

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Chapter 1

Introduction

The incompressible Navier Stokes (NS) equation and the Einstein equations are probably the most famous and well studied nonlinear differential equations in the mathematical physics. A lot of interesting phenomena such as turbulence, black holes, Big Bang, e.t.c. are connected to these equations. Therefore, the idea to relate these systems is very promising and was proposed back in 70's in the context of the membrane paradigm. The fluid/gravity relation reappeared in the sting theory context as a particular regime of the AdS/CFT correspondence. And each time the correspondence was proposed it immediately lead to interesting theorems - the black hole thermodynamics in case of the membrane paradigm and the famous viscosity to entropy bound in context of AdS/CFT.

The AdS/CFT and membrane paradigm approaches rely on different principles and yet they share some common results like viscosity to entropy ratio. In our research we tried to understand what are important ingredients to construct a relation between the NS equation and gravity so that we can explain the similarity of the results from different approaches. While pursuing this goal we have discovered the interesting feature of the fluid-dual metric being algebraically special. The more detailed look at algebraically special metrics allowed us to propose a new link between the fluid systems and special solutions to Einstein equations.

1.1 Outline

In the rest of this chapter we are going to briefly describe the fluid and gravity system(s) that we are going to connect and outline our construction for the fluid/gravity correspondence. In the chapter 2 we are providing a historical review the subject and describe the starting point of our research. In chapter 3 we discuss the linearized version of the correspondence for the most general background, while chapter 4 shows that all results can be generalized to the nonlinear theory using the hydrodynamic expansion. In chapter 5 we propose to use algebraically special metrics in order to construct dual fluid solutions. The possible generalization of our cMinkowskionstruction to the case of Magnetohydrodynamics/Einstein-Maxwell is discussed in chapter 6. The last chapter is used to discuss possible applications of our results in the search for exact solutions and understanding of turbulence.

1.2 Fluid Side

The "fluid"side of the fluid/gravity correspondence typically has a very broad interpretation, which includes such well known examples as incompressible fluids, relativistic fluids, magnetohydrodynamics, forced fluids, fluids in curved spaces. We want to present a brief introduction to the fluid dynamics and introduce useful formulas and notations. The Navier-Stokes equation [1] in $p+1$ dimensions is used to describe the motion of the viscous fluids

$$\rho(\partial_t v_i + v^j\partial_j v_i) = -\partial_i P + \frac{1}{2}\nu\partial^j(\partial_i v_j + \partial_j v_i) + \mu\partial_i\partial_k v^k + f_i, \quad i = 1, ..., p, \tag{1.1}$$

where $v_i(x,t)$ is a fluid velocity, $P(x,t)$ - fluid pressure, $\rho(x,t)$ is a fluid density and $f_i(x,t)$ is an external force. Usually, we can neglect the coordinate dependence for the shear viscosity η and the bulk viscosity μ, so they serve as a parameters in the equation. In particular the case $\nu = \mu = 0$ is called an ideal fluid. In order to get a closed system of equations the equation (1.1) is supplied with the mass conservation

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \tag{1.2}$$

and the equation of state

$$F(P,\rho) = 0. \tag{1.3}$$

Equations (1.1) and (1.2) at the absence of the external force f_i can be written as a conservation of the certain symmetric tensor with respect to the flat Minkowski metric $ds^2 = -dt^2 + dx_i^2$:

$$T_{tt} = \rho, \quad T_{ti} = -\rho v_i, \quad T_{ij} = \rho v_i v_j + P\delta_{ij} - \nu(\partial_i v_j + \partial_j v_i) - \mu \partial^k v_k \delta_{ij}. \tag{1.4}$$

On the linearized level there are two types of perturbative solutions to the fluid equations (1.1, 1.2, 1.3) in the form of sound modes and shear modes. Small perturbations of density $\delta\rho$ in an ideal fluid propagate as a sound waves and can be described in terms of the following equations

$$\partial_t \delta\rho + \rho_0 \partial_i v^i = 0, \quad \rho_0 \partial_t v_i + \partial_i \delta P = 0, \quad \delta P \partial_P F + \delta\rho \partial_\rho F = 0. \tag{1.5}$$

If we introduce $c^2 = \partial_\rho F / \partial_P F$ then the whole system is described by the wave equation

$$\partial_t^2 \delta\rho - c^2 \partial^2 \delta\rho = 0, \tag{1.6}$$

with c being the velocity of the sound. Note that in this case the fluid velocity v_i is a pure gradient, i.e. has only longitudinal component. The nonzero viscosity would add a dissipative term to the linear dispersion relation of the sound mode.

If the typical fluid velocity is much smaller then the velocity of sound, we can set $\rho = \rho_0$ and turn the mass conservation equation into incompressibility equation:

$$\partial_i v^i = 0, \tag{1.7}$$

while the Navier - Stokes equation (1.1) simplifies into:

$$\partial_t v_i + v^j \partial_j v_i + \partial_i P - \eta \partial^2 v_i = 0 \tag{1.8}$$

where we introduced the rescaled pressure $P \to \frac{P}{\rho_0}$ and the kinematic viscosity $\eta = \frac{\nu}{\rho_0}$. The viscosity controls the energy dissipation:

$$\partial_t \frac{\rho v^2}{2} = -\partial_i J^i - \frac{\eta}{2}(\partial_i v_j + \partial_j v_i)^2. \tag{1.9}$$

The viscosity η is a dimensionfull quantity and therefore can be scaled to $\eta = 1$ by suitable units choice.[1] Often the initial data or the fluid motion itself can be described in terms of the some velocity scale u and length scale L. For example, for the simple flow of the fluid in a round pipe, L is the radius of the pipe and u is a mean velocity. Out of three quantities η, u, L we can construct a dimensionless ratio, called the Reynolds number

$$Re = \frac{uL}{\eta}. \tag{1.10}$$

The Reynolds number is a good characteristic of the flow, i.e. flows that have the same Reynolds numbers can be obtained from one other by simple change of units for v and x. Interestingly, the large Reynolds numbers typically describe the turbulent fluid flows.[2]

[1] In the later chapters we will often set units so that the kinematic viscosity is just a number.

[2] We will have some discussion of the turbulence in chapter 7

Most of our duality discussions will be concentrated on the so called incompressible Navier - Stokes system on the fluid side

$$\partial_i v^i = 0,$$
$$\partial_t v_i + v^j \partial_j v_i + \partial_i P - \eta \partial^2 v_i = 0, \tag{1.11}$$

which we will be referring to as the NS equation. Our choice of particular equation on a the fluid side, the NS equation, is probably the most famous and most studied equation in mathematical physics. Therefore our project of establishing a map between it and some gravity setup is supplied with huge amount of known results for this equation. Certainly, there are many more other fluid systems with interesting physical applications and interesting dynamics and we will touch some of them in present work.

1.3 Gravity side

Recent progress in String theory introduced various modifications of gravitational theories by higher derivative terms, mass terms, e.t.c. Nevertheless the ordinary Einstein equations remain the most well-studied gravity equation with many known exact solutions, numerical simulations, and global properties for possible solutions. This gives us enough motivation to restrict our consideration to the case of Einstein equations with absent or simple matter stress tensor. n particular we will consider $(-, +, ..., +)$ signature metrics $g_{\mu\nu}$ in $p + 2$ dimensions that obey the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{1.12}$$

where $R_{\mu\nu}$ is the Ricci tensor. Most of our results also hold in a presence of the cosmological constant. In section 6 we are going to describe a simple generalization of the fluid/gravity correspondence for the Einstein-Maxwell equations.

1.4 Fluid/Gravity correspondence

There are two key observations which allow us to formulate the fluid/gravity correspondence. The first one is the observation that the NS equation can be written in the form of the covariant conservation of the following symmetric tensor (1.4). The second one is the existence of the conserved symmetric tensor for any solution to the Einstein equations. OnIn particular if the metric $g_{\mu\nu}$ satisfies Einstein equation then we can construct a symmetric conserved stress tensor for any codimension one hypersurface. This tensor is often called the Brown - York stress tensor and is defined via induced metric h_{ab} and extrinsic curvature K_{ab}

$$8\pi G t_{ab} \equiv h_{ab}K - K_{ab}. \tag{1.13}$$

Covariant conservation of t_{ab} is the same as one of the components of the Einstein equation. In order to set this tensor equal to the NS fluid stress tensor we need to impose some further condition to restrict the number of independent components. The general symmetric tensor in $p+1$ dimensions has $\frac{(p+1)(p+2)}{2}$ independent components while the fluid one has only $p+1$ (velocity vector and scalar pressure).

There are at least two ways to achieve this. We can consider a subset of metric that are produced by the vector gravitational perturbations which manifestly ensures that all gravitational tensors are parametrized by the single vector function. This approach is very natural in context of holography and quasinormal modes. We will discuss this approach in chapter 4 in great details. Second approach is related to the Petrov classification for metrics in higher dimensions and the fact that a particular Petrov type requirement provides just enough extra constrains to fix all the Brown-York tensor components in terms of the single vector and several scalars. The explicit constructions are provided in chapter 5.

Chapter 2

Historical Review

> There is no way to think up an original and extraordinary design - it can only come as a result of pursuing a given task. In the same way running down a list of words is different from making a narrative.
>
> Artemy Lebedev

The fluid/gravity correspondence has a very long and rich history. The first relations were proposed in 70's in context of the black hole horizon dynamics. In 2000's fluid/gravity duality emerge in context of the special sub sector of the AdS/CFT correspondence and lead to many interesting discoveries such as viscosity to entropy bound. The duality also become a powerful computational tool for a growing numerical simulations of the quark-gluon plasma dynamics.

In this section we want to touch some of key ideas and interesting results from the fortyyear history of the fluid/gravity correspondence. Due to the lack of space and time we cannot mention everyone who worked on this subject, so I apologize to the authors who will not be mentioned in this historical review. We will use the chronological order in this section as well as in the following sections since it seems suitable for PhD thesis format, so that the last chapter will include various follow-ups and generalization of our works.

2.1 The Membrane paradigm

The growing interest in black hole dynamics in 70's, complexity of Einstein equations and some lack in computational power for numerical simulations led to formulation of so called membrane paradigm by Damour, Price, Thorne, Hartle, Hawaking and others [2–11]. The idea was to replace a black hole horizon with a viscous fluid and use fluid dynamics methods to study the horizon evolution in black hole collision processes etc.

The math behind the membrane paradigm is the null surface evolution by the Einstein equation. There is a nice modern review by Gourgoulhon [12, 13]. The geometry of the p - dimensional null surface $\mathcal{H}$ is captured by induced metric γ_{ij}, the symmetric traceless shear tensor σ_{ij}, normal fundamental one form Ω_i, surface gravity κ and expansion rate θ. The $G_{\ell i}$ component of the Einstein equation is

$$\mathcal{L}_\ell \Omega_i + \theta\Omega_i + \partial_i\kappa + \frac{1}{2}\partial_i\theta - \nabla^j\sigma_{ij} = 0, \tag{2.1}$$

where ℓ is a null normal to the null surface $\mathcal{H}$. The equation (2.1) is often referred in a literature as Damour-Navier-Stokes equation. It has a structure very similar to the NS equation (1.11) if we make identifications

$$P \sim (\kappa - \frac{1}{2}v^2), \quad \Omega_i \sim v_i, \quad \ell = \partial_t + v^i\partial_i, \quad \sigma_{ij} + \frac{1}{2}\theta\gamma_{ij} \sim \frac{1}{2}(\partial_i v_j + \partial_j v_i), \tag{2.2}$$

and assume that the expansion rate is very small

$$\theta \sim \partial_i v^i \approx 0. \tag{2.3}$$

Another interesting component is the $G_{\ell\ell}$ one

$$\mathcal{L}_\ell \theta + \theta^2 - \kappa\theta = \frac{1}{2}\theta^2 - \sigma_{ij}\sigma^{ij}. \tag{2.4}$$

It has a structure of the energy balance equation. For the small and stationary expansion rate the equation simplifies

$$\kappa\theta \approx \sigma_{ij}\sigma^{ij}. \tag{2.5}$$

The expansion rate is related to the change of the horizon area element $\mathcal{L}_\ell \ln \Delta A$ while the area is related to the black hole entropy

$$S = \frac{A}{4G}. \tag{2.6}$$

The surface gravity is related to the black hole temperature and further related to the black hole energy change ΔE

$$\kappa\theta\Delta A = 8\pi G \mathcal{L}_\ell \Delta E. \tag{2.7}$$

all of above leads to very interesting interpretation of the (2.4) in the form of

$$\partial_t \Delta E \approx \frac{1}{8\pi G}\kappa\theta\Delta A = \frac{1}{8\pi G}\sigma_{ij}\sigma^{ij}\Delta A = \frac{1}{2}\eta(\partial_i v_j + \partial_j v_i)^2 \Delta A. \tag{2.8}$$

Which is the well known viscous fluid dissipation equation and the viscosity is

$$\eta = \frac{1}{16\pi G}. \tag{2.9}$$

The viscosity is a dimensionfull quantity, but the Bekenstein-Hawking formula and the membrane interpretation of the fluid allowed us to introduce a universal entropy density

$$s = \frac{1}{4G}, \tag{2.10}$$

which can be used to describe a dimensionless ratio

$$\frac{\eta}{s} = \frac{1}{4\pi}, \tag{2.11}$$

which become quite famous in early 2000's and is refered as viscosity to entropy ratio. The specific value of this ratio is subject to a certain bound, which in turn is being saturated by the membrane paradigm value.

The membrane paradigm approach provides a nice physical interpretation of the black hole horizon as a fluid system. However the Damour-Navier-Stokes equation (2.1) is not a NS system since for a general black hole geometry Ω_i and σ_{ij} are independent variables. The equation is essentially a proper null limit of the Brown-York tensor conservation from the previous chapter, so we need to provide some additional input to reduce the number of independent components. Similarly without knowing and explicit relation between Ω_i and σ_{ij} we cannot specify the value of viscosity to entropy ratio. The fact that is is exactly saturates the bound is due to the great intuition of the membrane paradigm authors.

2.2 Quasinormal modes

The next step important step in the fluid/gravity development was done in context of the AdS/CFT correspondence [14–16]. Relativistic fluid appeared as an effective description of the boundary CFT for large temperature and finite density. In the most studied case of the AdS_5/CFT_4 correspondence the field theory's stress tensor admits an expansion

$$T_{\mu\nu} = T^4 T^{perfect}_{\mu\nu} + T^3 T^{viscous}_{\mu\nu} + \mathcal{O}(T^2), \tag{2.12}$$

where $T^{perfect}$ is the perfect fluid stress tensor

$$T^{perfect}_{\mu\nu} = (\mathcal{E} + P) u_\mu u_\nu + P g_{\mu\nu}, \tag{2.13}$$

where u_μ is a fluid four velocity, $P, \mathcal{E}$ are pressure and density. $T^{viscous}$ is the viscous term

$$T^{viscous}_{\mu\nu} = \eta\sigma_{\mu\nu}, \tag{2.14}$$

where η is fluid viscosity, $\sigma_{\mu\nu}$ is the shear tensor. The dimensionless expansion parameter is the $\frac{\partial}{T}$, where ∂ stands for a derivatives of the fluid variables. Such expansion is very natural for both fluid dynamics and CFT therefore lead to many interesting results.

The dual gravity description is a perturbed AdS_5 black brane solution

$$ds^2 = -r^2(1 - \tfrac{b^4}{r^4})du^2 + 2dudr + r^2 dx_i^2 + ..., \tag{2.15}$$

where T is the the CFT temperature. AdS/CFT correspondence conjectures that the gravitational dynamics is related to the CFT dynamics at hypersurface Σ_c at large fixed $r = r_c, r_c \to \infty$.

Policastro, Strainets and Son (PSS) [17,18] proposed an interesting test for the AdS_5/CFT_4 correspondence for the setup described above. The viscous fluid admits so called shear perturbation mode

$$\omega = -\tfrac{i\eta}{P+\mathcal{E}}k^2, \quad \omega, k \ll T, \tag{2.16}$$

where k is spatial momentum and ω is the frequency.

The $\omega \sim k^2$ scaling follows from the diffusion equation, similar to the linearized NS equation. This mode is special because it does not contain the leading k^1 term, therefore the search for a dual gravity mode is an interesting question. The mode is decaying so the dual mode should be a quasinormal mode of the black brane geometry. Since we are looking for a linearized perturbation we can use the $SO(3)$ symmetry to classify the possible gravitational perturbations into tensor, vector and scalar perturbations.[1] Careful consideration shows that

[1] see [19] for nice review of different perturbation types.

the tensor and scalar modes at small ω, k obey $\omega \sim k$ dispersion, while the vector modes have $\omega \sim k^2$.

In the coordinates (2.15) the vector perturbation has the following form

$$ds^2 = -r^2\left(1-\frac{b^4}{r^4}\right)d\tau^2 + 2dudr + r^2 dx_i^2 \tag{2.17}$$

$$+ 2f_1(r)v_i dx^i d\tau + r^2 f_2(r)(\partial_i v_j + \partial_j v_i)dx^i dx^j + ... \tag{2.18}$$

The Einstein equations for this metric ansatz can be turned into a simple differential equation in r-variable[2]. The ingoing boundary conditions on the black brane horizon and the Dirichlet boundary conditions on Σ_c allows to solve for dispersion law.

$$\omega = -iDk^2, \quad D = \tfrac{1}{4\pi T}. \tag{2.19}$$

For the relativistic fluid there is a dimension less viscosity to entropy ratio η/s which was computed by PSS for the 5d black brane example

$$\tfrac{\eta}{s} = \tfrac{1}{4\pi}. \tag{2.20}$$

Furthermore the follow-up works [20–24] showed that this ratio is rather universal for the string theory - relevant gravitate theories. Later papers [25–28] showed that universality holds for any two derivative theory of gravity in AdS, while it is modified by the extra derivative terms [29–36]. Moreover the ratio exactly matches the membrane paradigm value, which derivation relied on rather general properties of the gravitational solution. The fact that the values match, while being computed using different approaches was essentially a starting point of our fluid/gravity research. In our first paper [37] we addressed a question of defining the viscosity to entropy ratio for a general geometry and arbitrary hypersurface. We present the details of these computations in chapter 3.

[2] The details of the similar procedure for the more general metric is described in chapter 3.

2.3 Minwalla's approach

PSS work on relating the linearized modes in the relativistic fluid dynamics and the quasinormal modes for AdS_5 solutions naturally lead to the same question for nonlinear theories. Shiraz Minwalla and many of his collaborators [38–45] proposed a systematic perturbation theory for both fluid dynamics and Einstein equations in AdS.

The base space-time for perturbative expansion is the boosted black brane metric in 5d

$$ds^2 = -2u_\mu dx^\mu dr + (b^2 r)^{-2} u_\mu u_\nu dx^\mu dx^\nu + r^2 \eta_{\mu\nu} dx^\mu dx^\nu, \tag{2.21}$$

with u_μ being four velocities, which can be parametrized

$$u^\mu = \frac{1}{\sqrt{1-v^2}}(1, v^i), \tag{2.22}$$

and b is related to the temperature via

$$T = \frac{1}{\pi b}. \tag{2.23}$$

For any constant v^i, b the metric (2.21) solves the Einstein equations with cosmological constant. On the CFT side this metric corresponds to the ideal relativistic fluid with a stress tensor

$$T^0_{\mu\nu} = (\pi T)^4 (\eta_{\mu\nu} + 4u_\mu u_\mu). \tag{2.24}$$

The mapping is provided by the suitably renormalized Brown - York stress tensor at asymptotic boundary of AdS at $r = \infty$. The key observation is to replace b, v_i by a slow varying functions $b(x_\mu), v_i(x_\mu)$. The new metric will not be a solution of the Einstein equations because of the nontrivial derivatives of $b(x_\mu)$ and $v_i(x_\mu)$, but we can add some additional terms to the metric that contain derivatives to cancel them. The procedure is well defined because of the observation that the derivatives action of the $\ln b$ and v_i always appear together with the factor of b. So the derivative terms that paper in the Einstein equations are suppressed by powers of b/L, where L is length scale of the variations. The background value of b is related to the black brane temperature so we naturally have a dimensionless expansion parameter $(TL)^{-1}$.

The first order correction to the metric results in the corresponding correction to the Brown-York stress tensor at infinity

$$T^{0+1}_{\mu\nu} = (\pi T)^4(\eta_{\mu\nu} + 4u_\mu u_\mu) - 2(\pi T)^3\sigma_{\mu\nu}, \tag{2.25}$$

so that the first order CFT hydrodynamics is just a viscous fluid. Moreover we can calculate the viscosity to entropy ratio for such fluid

$$\tfrac{\eta}{s} = \tfrac{1}{4\pi}. \tag{2.26}$$

The first order deformation result is in perfect agreement with PSS computations for the linearized theory. The similar analysis was performed in the case of nonrelativistic fluids [46, 47].

2.4 Questions

In previous sections we briefly summarized the most important results in fluid/gravity prior to our works. Our starting point was to fill the gaps and answer the following questions

- Why is the viscosity entropy ratio for both membrane paradigm and AdS/CFT approach?
- Is AdS background is necessary ingredient for the fluid/gravity correspondence?

- Is it possible to get a fluid equations for a general hypersurface (not being horizon or asymptotic AdS boundary)?
- What are the dual fluids for some simple exact black hole solutions?

We manage to succeed in answering first three questions and proposed some approach that can be used to address the last one. In the process we have discovered new interesting features in the fluid/gravity correspondence, which we will discuss in the following chapters.

Chapter 3

Linearized theory

On the face of it, the Damour and PSS computations are very different. However, there are strong indications that they are related. Firstly, they both relate a theory of gravity to a "dual" fluid theory living in one fewer dimension; without the radial direction. Secondly, both approaches lead to the same numerical ratio for $\frac{\eta}{s}$. An important difference is that the Damour calculation is performed at the black hole horizon $r = r_h$, while the PSS calculation is performed at spatial infinity $r = \infty$. Both the basic relation between redshift and radius[1] and, in the special context of string theory, results from AdS/CFT suggest that from the point of view of the fluid theory (which does not have an r coordinate) changing r is equivalent to renormalization group (RG) flow. Hence one expects the Damour calculation to be related to the PSS calculation by some kind of RG flow into the IR. This view is advocated in [22, 26, 28, 36, 47–50].[2] In order to verify this expectation, one must first define what one

[1] Radial transformations are referred to as renormalization already in [9]

[2] These references typically study RG flow by looking at the r-dependence of correlators whose boundary conditions are imposed in an asymptotically AdS region. This corresponds to choosing a specific UV completion. In this paper we will formulate the problem in a way that does not involve such a choice.

means by the gravity and fluid theory associated to finite r. In this paper, among other things, we propose a precise definition of the finite r theory and show that the expectation is indeed realized.

The basic idea is to introduce a cutoff surface Σ_c at some fixed radius $r = r_c$ outside the black hole or brane. We then impose ingoing boundary conditions at the horizon and fix the induced metric on Σ_c. These boundary conditions do not fully specify a solution. The problem is then to identify the remaining internal degrees of freedom and describe their dynamics. We solve this to linear order in the internal fluctuations (in appropriate expansions) and show they correspond to those of a fluid. A formula is derived for the diffusion rate and other hydrodynamic quantities, which generically run as a function of the cutoff r_c. We hope that it is possible to extend our approach beyond the next-to-leading order considered here, but we defer that problem to future work.

This reformulation of fluid/gravity duality is the analog - or holographic dual - of Wilson's reformulation of quantum field theory. Wilson did not insist on an ultraviolet completion of quantum field theory, and we do not insist on an asymptotically AdS region of the geometry. Specifying the couplings at the Wilsonian cutoff Λ_W is the analog of specifying the boundary conditions for the induced metric (and other fields if present) on Σ_c . If we scatter fields at energies below Λ_W, we needn't know anything about the theory at energies above Λ_W. Similarly if we disturb a black hole by throwing something at it from the radius r_c, we needn't know anything about the geometry outside r_c.[3] One advantage of the Wilsonian approach is that a much broader class of theories can be discussed.

In addition to providing new methods of computation and broadening the space of ap-

[3]In this analogy, large r_c corresponds to large Λ_W, but a precise functional relation will not be found herein.

plications, several new qualitative insights are gained in this approach. As the cutoff is taken to the horizon ($r_c \to r_h$), the relevant geometry is simply Rindler space, and the transport coefficients all approach simply computable universal values which are largely insensitive to details of the geometry or matter couplings.[4] In particular $\frac{\eta}{s} \to \frac{1}{4\pi}$. Moreover the fluctuations dissipate according to the linearized Navier-Stokes equations, with *no higher-derivative corrections*. Hence our near horizon scaling is the geometric version of the low-velocity scaling in which fluids are governed by (as it turns out incompressible) Navier-Stokes. At any finite $r_c > r_h$ there are infinitely many higher derivative corrections to the Navier-Stokes dispersion relation, and one computes only the leading term at long wavelengths. The leading dispersion constant in general runs and does not take a universal value at radial infinity. If we specialize to an asymptotically AdS black brane and take $r_c \to \infty$, our computations are all in manifest agreement with the usual AdS/CFT definitions of the transport coefficients. All of this supports the picture that Damour was computing in the IR of the dual fluid theory while PSS were computing in the UV. The extra ingredients required for the PSS calculation are the extra ingredients needed to specify a theory all the way up to the UV, while the universality of $\frac{\eta}{s}$ is a characteristic of the IR fixed point and needs only the IR near-horizon Rindler space.

Interestingly enough, in contrast to the generic transport coefficient, in the classical gravity limit the particular ratio $\frac{\eta}{s}$ typically does *not* run and equals $\frac{1}{4\pi}$ everywhere. This is why the UV and IR PSS and Damour calculations agree for this quantity. Although still partially mysterious to us, we show that this RG-flow invariance stems from the first law of thermodynamics applied to the radial flow, together with the fact that, in the classical

[4]This remains true as long as the linearized gravity fluctuations are governed by the linearized Einstein equation, which is not the case with higher derivative corrections.

gravity limit, there is no entropy except horizon entropy and the flow is therefore isentropic. This will not be the case when quantum corrections are included, as there is then entropy in the gas of Hawking radiation as well as entanglement entropy across Σ_c. Therefore we expect $\frac{\eta}{s}$ to run at the quantum level.

3.1 Outline

This chapter is organized as follows. In section 2 we write down the most general $p+2$-dimensional general geometry which, on symmetry grounds, could serve as a holographic dual to a fluid in $p+1$ dimensions. Explicit expressions are given for the asymptotically flat and asymptotically AdS black branes with and without charges to serve as illustrative examples.

Section 3 treats the case of an electromagnetic field propagating on these geometries as a simple warm up. Dirichlet (ingoing) boundary conditions are imposed at the cutoff surface $r = r_c$ (horizon $r = r_h$). It is then shown in a long-wavelength expansion that the remaining dynamical modes are described by a charge density which evolves according to Fick's law in $p+1$ flat dimensions. The diffusion constant is given by a line integral of certain metric coefficients from r_h to r_c, and "runs" as r_c is varied. If the cutoff r_c is taken to r_h, no long wavelength expansion is needed and the Fick law becomes exact. Moreover it is shown in this limit that, after carefully normalizing by the divergent local Unruh temperature, the diffusion constant approaches a universal constant determined by the properties of Rindler space.

In section 4 the analysis is adapted to linearized gravity fluctuations. After fixing the cutoff and horizon boundary conditions, the vector or shear modes are shown to obey the linearized Navier-Stokes equation in a long wavelength expansion, and a running formula for the diffusion constant is derived. As the cutoff is taken to the horizon, the linearized Navier-Stokes equation becomes exact and the constant is shown to approach the universal Damour value. Some special features of the RG flow for the gravitational case are also discussed. Tensor modes are shown to have no dynamics in the appropriate limit, while there is a dynamical "sound" mode in the scalar sector. Very interestingly, the effective speed of sound goes to infinity and hence the sound mode decouples as the cutoff is taken to the horizon. This means the fluid is becoming incompressible. Specific examples of the charged and neutral AdS black branes and asymptotically flat S^3-reduced NS5 branes are worked out in detail.

In section 5 we introduce the Brown-York stress tensor t_{ab} on the cutoff surface. Prior to this point only equations of motion have been used so entropy, viscosity, energy and pressure (which depend on the normalization of the action) could not be discussed. We show that t_{ab} not only is conserved with our boundary conditions but takes the form of a fluid stress tensor (to linear order). We compute the thermodynamic quantities in terms of the spacetime geometry.

In section 6 we compute the viscosity to entropy ratio $\frac{\eta}{s}$ and show that, under rather general assumptions, the radial RG evolution equations imply it is cutoff independent and equal to $\frac{1}{4\pi}$ for Einstein gravity. It is shown that these radial equations are nothing but - in the fluid picture - the first law of thermodynamics for isentropic variations in disguise. The radial flow is isentropic because in classical gravity there is no entropy outside the black hole. It is accordingly suggested that the RG-invariance of $\frac{\eta}{s}$ will be violated by quantum gravity corrections.

3.2 Background geometry

3.2.1 The general case

In this paper we are interested in studying the dynamics of fluids in p flat space dimensions and one flat time dimension. The holographic dual of such a fluid in its ground state should be a $p+2$-dimensional spacetime geometry, with isometries generating the Euclidean group of p-dimensional rotations/translations plus time translations. The corresponding line element can be written in the form

$$ds^2_{p+2} = -h(r)d\tau^2 + 2d\tau dr + e^{2t(r)}dx^i dx_i, \tag{3.1}$$

where the index $i = 1, ...p$ here and hereafter is raised and lowered with δ_{ij}. Lines of constant

tau and varying r are null. We consider the case where there is a horizon r_h at which

$$h(r_h) = 0, \tag{3.2}$$

and $h(r)$ is positive for $r > r_h$. Lines of constant $r = r_h$ and varying τ are the null generators of the future horizon, while those of constant $r > r_h$ and varying τ are timelike and accelerated. For convenience we choose the scaling of the spatial x^i coordinate so that

$$t(r_h) = 0. \tag{3.3}$$

A special role will be played by the "cutoff" surfaces Σ_c of constant $r = r_c > r_h$. The induced metric on such a surface is flat $p + 1$-dimensional Minkowski space

$$ds^2_{p+1} = -h\left(d\tau - \frac{dr}{h}\right)^2 + e^{2t}dx^i dx_i, \tag{3.4}$$

with τ the time coordinate. We will sometimes collectively denote the Minkowskian coordinates by

$$x^a \sim (x^i, \tau), \qquad a = 0, \cdots p. \tag{3.5}$$

It is convenient to introduce proper intrinsic coordinates on Σ_c

$$x_c^0 = \sqrt{h(r_c)}\tau, \quad x_c^i = e^{t(r_c)} x^i. \tag{3.6}$$

The advantage of these coordinates is that the induced metric is simply

$$ds_{p+1}^2 = \eta_{ab} dx_c^a dx_c^b, \tag{3.7}$$

so that they directly measure proper distances on Σ_c. Full bulk coordinates will be denoted

$$x^\mu \sim (x^i, \tau, r), \qquad \mu = 0, \cdots p+1. \tag{3.8}$$

We denote by ℓ^μ the normal satisfying

$$\ell^\mu \partial_\mu = (\partial_\tau + h\partial_r), \qquad \ell^2 = h. \tag{3.9}$$

At $r = r_h$, ℓ is null, normal and tangent to the future horizon.

3.2.2 Some special cases

Here we collect some specific examples which will be used as illustrations in the text. (5.14) of course reduces to flat space for $h = t = 0$. Another useful way to write flat space is in "ingoing Rindler" form[5]

$$\begin{gathered} h(r) = r, \qquad t = 0, \\ ds_R^2 = -r d\tau^2 + 2 d\tau dr + dx^i dx_i. \end{gathered} \tag{3.10}$$

[5]Writing $\tau = 2\ln t^+, \quad r = -t^+ t^-$, the 2D part of the metric becomes $-4dt^+ dt^-$.

Observers at fixed $r > 0$ and x^i are then Rindler observers.

The asymptotically AdS$_{p+2}$ black p-brane solutions are

$$\begin{aligned} h &= \frac{r^2}{R^2}\Big(1 - \frac{r_h^{p+1}}{r^{p+1}}\Big), \quad e^t = \frac{r}{r_h}, \\ ds^2_{BB} &= -\frac{r^2}{R^2}\Big(1 - \frac{r_h^{p+1}}{r^{p+1}}\Big)d\tau^2 + 2d\tau dr + \frac{r^2}{r_h^2}dx_i dx^i. \end{aligned} \tag{3.11}$$

Rindler space (3.10) is a limit of the black brane geometry (3.11). To see this define

$$\begin{aligned} r' &= \frac{R^2(r - r_h)}{(p+1)r_h}, \\ \tau' &= \frac{(p+1)r_h}{R^2}\tau. \end{aligned} \tag{3.12}$$

The horizon is then at $r' \to 0$ near which

$$ds^2_{BB} = \left(-r'd\tau'^2 + 2d\tau'dr' + dx^i dx_i\right)\left(1 + \mathcal{O}(\frac{r'}{R^2})\right). \tag{3.13}$$

Hence Rindler space is both the near-horizon and $R \to \infty$ limits of the black brane.

If we add a $U(1)$ gauge field and charge density Q to the black brane, the metric and gauge field A are

$$\begin{aligned} h &= \frac{r^2}{R^2}\left(1 - \left(1 + \alpha Q^2\right)\frac{r_h^{p+1}}{r^{p+1}} + \alpha Q^2 \frac{r_h^{2p}}{r^{2p}}\right), \quad e^t = \frac{r}{r_h}, \\ A &= \frac{Q r_h}{p-1}\left(1 - \frac{r_h^{p-1}}{r^{p-1}}\right) d\tau. \end{aligned} \tag{3.14}$$

Here $\alpha = \frac{R^2 8\pi G}{p(p-1)}$, and we have set the electromagnetic coupling constant to one.

A specific example with no AdS region is provided by the well-studied asymptotically flat NS5 brane (we omit the case of general p for brevity). This is a solution of ten-dimensional supergravity with a three-form field strength threading an S^3 which surrounds the brane. This ten-dimensional geometry of course is not of the form (5.14). However if we Kaluza-Klein reduce to 7 dimensions on the S^3, then it does take this form. In the 7-dimensional

Einstein frame and coordinates (5.14), the metric is

$$h(r) = y^{6/5} \left(1 + \frac{L^2}{y^2}\right)^{1/5} \left(1 - \frac{y_h^2}{y^2}\right), \quad e^{2t(r)} = \frac{y^{6/5}}{y_h^{6/5}} \left(1 + \frac{L^2}{y^2}\right)^{1/5} \left(1 + \frac{L^2}{y_h^2}\right)^{-1/5} \tag{3.15}$$

Here $y(r)$ is the solution of

$$r = \int^{y} dy' \, y'^{6/5} \left(1 + \frac{L^2}{y'^2}\right)^{7/10} . \tag{3.16}$$

The right hand side is a hypergeometric function.

We wish to stress that our approach applies to geometries of the general form (5.14) and is not tied to the above specific examples. Other interesting examples include the proposed holographic duals to superconductors [51], for which the metric cannot in general be found analytically, or other cases ([52,53] to mention a few) which are not asymptotically AdS and correspond to systems which are not conformally invariant in the UV.

3.3 Electromagnetic warmup

In this section we warm up to the gravity problem by considering the conceptually similar, but mathematically simpler, problem of an electromagnetic field F propagating in the geometry (5.14).

3.3.1 The setup

Our first step is to introduce a cutoff surface Σ_c outside the horizon in the general geometry (5.14)

$$\Sigma_c : \qquad r = r_c > r_h, \tag{3.17}$$

with coordinates

$$x^a \sim (x^i, \tau). \tag{3.18}$$

The induced metric on Σ_c is flat and given by (3.4) with $r = r_c$. We wish to study the dynamics of an electromagnetic field F within the region

$$r_h \leq r \leq r_c. \tag{3.19}$$

This requires boundary conditions at both r_c and r_h. Since r_h is a black hole horizon, we impose ingoing boundary conditions there. At r_c, a natural Dirichlet-like choice is to fix the components of the field strength tangent to Σ_c

$$F_{ab}(x^e, r_c) = f_{ab}(x^e) \qquad a, b, e = 0, ...p. \tag{3.20}$$

We view the fs as the parameters defining the cutoff theory, and Σ_c as the place where experiments are set up and measurements made which probe the entire region (3.19) below the cutoff. Fixing a radius where experiments are performed is dual, in the fluid picture, to fixing the scale at which experiments are performed.

The boundary conditions at r_c and r_h do not uniquely specify a solution of the Maxwell equations for F. The problem is to describe the remaining dynamical degrees of freedom. We will see that, in the limits we consider, they are described by a single function $q(x^a)$ which obeys a simple diffusion equation on p+1-dimensional Minkowski space. q can be thought of as a charge density on the horizon or, by a simple rescaling, as a charge density at the cutoff q_c. We will compute the diffusivity and also see that the data f_{ab} specifying the boundary conditions at Σ_c function as a source for the charge density. This enables cutoff observers to probe the dynamics of the charge density.

Let us now turn to the details of this description.

3.3.2 Equations and boundary conditions

We have a bulk gauge field with components $F_{\tau r},\ F_{ir},\ F_{i\tau},\ F_{ij}$. In terms of these the bulk Maxwell's equations may be written

$$r: \qquad e^{2t}\partial_\tau F_{\tau r} + \partial^i F_{\tau i} = h\partial^i F_{ir}, \tag{3.21}$$

$$i: \quad \partial_\tau F_{ir} + ((2-p)t' - \phi')F_{\tau i} - \partial_r F_{\tau i} + h' F_{ir} = -h\partial_r F_{ir} + h((2-p)t' - \phi')F_{ir} + e^{-2t}\partial^j F_{ji}, \tag{3.22}$$

$$\tau: \qquad \partial_r F_{\tau r} + (pt' + \phi')F_{\hat{v}r} + e^{-2t}\partial^i F_{ir} = 0, \tag{3.23}$$

where $\partial^i = \delta^{ij}\partial_j$ and for future utility we have allowed for a position-dependent gauge coupling $\frac{1}{g^2} = e^\phi$ normalized so that $g(r_h) = 1$ and $\phi(r_h) = 0$. In addition we will need the Bianchi identities

$$\partial_r F_{i\tau} = \partial_\tau F_{ir} - \partial_i F_{\tau r}, \tag{3.24}$$

$$\partial_r F_{ij} = \partial_i F_{rj} - \partial_j F_{ri}. \tag{3.25}$$

We wish to impose ingoing boundary conditions on the gauge field at the future horizon. As our coordinates are regular on this horizon, we require F to be regular there:[6]

$$F_{ri}(r_h) = \text{finite}. \tag{3.26}$$

The other data at the horizon are the horizon current and charge density, defined as[7]

$$j_i(x^a) \equiv F_{i\hat{v}}(x^a, r_h), \qquad q(x^a) \equiv F_{r\hat{v}}(x^a, r_h), \tag{3.27}$$

[6]We thank Stephen Green and Robert Wald for pointing out an error in the previous version of our paper.

[7]We may also write this in terms of the normal (3.9) as $(q, j_i)^a = F^{ab}\ell_b$.

together with $F_{ij}(x^a, r_h)$. Given the regularity condition, the Maxwell equation (3.21) at the horizon becomes current conservation

$$\partial_\tau q + \partial^i j_i = 0. \tag{3.28}$$

3.3.3 Long wavelength expansion

In this subsection we introduce a non-relativistic long-wavelength expansion which is suitable for studying hydrodynamics.

- *Solving the equations*

The general solution of the Maxwell equation cannot be found analytically in a general geometry of the form (5.14). To proceed further we consider a long-wavelength expansion parameterized by $\epsilon \to 0$. We take temporal and spatial derivatives to have the non-relativistic scaling

$$\partial_\tau \sim \epsilon^2, \qquad \partial_i \sim \epsilon. \tag{3.29}$$

The gauge field has the associated expansion

$$\begin{aligned} F_{ir} &= \epsilon\big(F^0_{ir} + \epsilon F^1_{ir} + \cdots\big), \\ F_{\tau r} &= \epsilon^2\big(F^0_{\tau r} + \epsilon F^1_{\tau r} + \cdots\big), \\ F_{ij} &= \epsilon^2\big(F^0_{ij} + \epsilon F^1_{ij} + \cdots\big), \\ F_{i\tau} &= \epsilon^3\big(F^0_{i\tau} + \epsilon F^1_{i\tau} + \cdots\big). \end{aligned} \tag{3.30}$$

We will solve for $F^0_{\mu\nu}(x^a, r)$ in terms of its value at the horizon $r = r_h$ by integrating the first order radial evolution equations (3.22) and (3.23) outward to $r > r_h$, and then demanding agreement with the boundary conditions (3.20) when the cutoff $r = r_c$ is reached. At lowest order in ϵ these equations are

$$i0: \qquad h' F^0_{ir} = -h\partial_r F^0_{ir} + h((2-p)t' - \phi')F^0_{ir}, \tag{3.31}$$

$$\tau 0: \qquad \partial_r F^0_{\tau r} + (pt' + \phi')F^0_{\tau r} + e^{-2t}\partial^i F^0_{ir} = 0. \tag{3.32}$$

There are no non-trivial solutions of (3.31) which obey the ingoing boundary condition (3.26). Therefore

$$F^0_{ir} = 0. \tag{3.33}$$

The general solution of the second equation (3.32) is then

$$F^0_{\tau r}(x^a, r) = -e^{-pt(r)-\phi(r)}q^0(x^a). \tag{3.34}$$

The leading order Bianchi identity

$$\partial_r F^0_{i\tau} = e^{-pt(r)-\phi(r)}\partial_i q^0 \tag{3.35}$$

then implies the leading term in $F_{i\hat{v}}$

$$F^0_{i\tau}(x^a, r) = -\int_r^{r_c} ds e^{-pt(s)-\phi(s)}\partial_i q^0(x^a) + f^0_{i\tau}(x^a), \tag{3.36}$$

where we have used the boundary conditions at the cutoff to determine the integration constants. Evaluated at the horizon $r = r_h$ (3.36) gives the Fick-Ohm law in $p+1$ dimensions

$$j^0_i = -D^{EM}_c \partial_i q^0 + f^0_{i\tau}, \tag{3.37}$$

with diffusivity given by the line integral

$$D^{EM}_c(r_c) = \int_{r_h}^{r_c} ds e^{-pt(s)-\phi(s)}. \tag{3.38}$$

Current conservation (3.28) then implies

$$\partial_\tau q^0 = D_c^{EM}\partial^2 q^0 - \partial^i f^0_{i\tau}. \tag{3.39}$$

In particular if we choose conducting boundary conditions so that the electric field vanishes at the cutoff we find Fick's second law

$$\partial_\tau q^0 = D_c^{EM}\partial^2 q^0. \tag{3.40}$$

Taking the Fourier transform of this equation, we see that the charge density propagates according to the dispersion relation

$$i\omega = D_c^{EM} k^2. \tag{3.41}$$

We still need to solve for $F^0_{ij}(x^a, r)$. The leading term in the Bianchi identity (3.25) is

$$\partial_r F^0_{ij} = 0. \tag{3.42}$$

The solution of this with the given boundary conditions is simply

$$F^0_{ij}(x^a, r) = f^0_{ij}(x^a). \tag{3.43}$$

Hence there are no dynamics associated with F^0_{ij}.

To leading order in ϵ, (3.33, 3.34, 3.36)and (3.43) comprise the most general solution of the Maxwell equation with the given boundary data. The solution is characterized by a single function in $p+1$ dimensions, the charge density $q^0(x^a)$, which obeys the dispersion relation (3.110). Away from the scaling limit $\epsilon = 0$, the dispersion relation (3.73) for q has corrections of order k^4.

Other parameterizations of the solution space are possible. One alternative is to take the normal component of the electric field at the cutoff, which is not fixed by the boundary condition there. This is simply related at leading order to the horizon charge density, according to (3.34), by [8]

$$q_c^0(x^a) \equiv e^{2t(r_c)} F^0_{r\tau}(x^a, r_c) = e^{-(p-2)t(r_c)-\phi(r_c)} q^0(x^a), \tag{3.44}$$

and hence obeys the same dispersion relation. This parameterization is more natural from our perspective as we want to think of measuring the dynamical modes and dispersion constants with experimental devices positioned on the cutoff surface Σ_c. The charge density can be sourced, according to (3.110), by turning on an electric field $f_{i\tau}$. Hence the dispersion constant can be measured by turning on a tangential electric field at the cutoff and watching how the normal component decays. We shall henceforth adopt q_c as our basic variable.

- *Normalizing the diffusivity*

As it stands, the value of D_c^{EM} (3.73) is not very meaningful because it is dimensionful and can be set to any value by a change of coordinates. We therefore introduce the proper frequency ω_c and proper momentum k_c conjugate to proper time and distance in the cutoff hypersurface $r = r_c$. These are

$$\omega_c = \frac{\omega}{\sqrt{h(r_c)}} \sim -i\frac{\partial}{\partial x_c^0}, \quad k_c^i = e^{-t(r_c)} k^i \sim i\frac{\partial}{\partial x_c^i}. \tag{3.45}$$

Observers at the cutoff equipped with a thermometer measure a non-zero temperature T_c.[9] To determine T_c, note that very near the horizon $r \to r_h$, observers with worldlines of fixed r and x^i are highly accelerated. They therefore detect a Rindler temperature which diverges as[10]

$$T_R = \frac{\partial_r h}{4\pi h^{1/2}} = \frac{\sqrt{h'(r_h)}}{4\pi\sqrt{r - r_h}} + \mathcal{O}\left(\sqrt{r - r_h}\right) \tag{3.46}$$

[8]We could of course have couched our entire argument in terms of q_c, but the derivation is then burdened by extra terms which cancel in the end.

[9]The formulation here naturally leads to a cutoff-dependent temperature. If we want to keep the temperature at the cutoff fixed as we change r_c, we would have to simultaneously move in the space of black-brane solutions.

[10]We have set $\hbar = 1$, otherwise it would appear on the right hand side of this equation.

for any smooth quantum state. For an equilibrium state, the temperature as a function of r is determined by the Tolman relation

$$T(r)\sqrt{h} = T_H = \text{constant}. \tag{3.47}$$

For an asymptotically flat spacetime, $h \to 1$ at infinity, and T_H is the Hawking temperature. The relation (3.47) together with the boundary condition (3.46) determines the local temperature at the cutoff to be

$$T_c \equiv T(r_c) = \frac{T_H}{\sqrt{h(r_c)}}, \qquad T_H = \frac{h'(r_h)}{4\pi}. \tag{3.48}$$

The quantity $\bar{D}_c^{EM}$ defined by

$$i\omega_c = \frac{\bar{D}_c^{EM}}{T_c} k_c^2, \tag{3.49}$$

then gives the diffusivity in units of the cutoff temperature. Our final expression for the coordinate-invariant, dimensionless diffusivity $\bar{D}_c^{EM}$ is

$$\bar{D}_c^{EM} = \frac{e^{2t(r_c)} h'(r_h)}{4\pi h(r_c)} \int_{r_h}^{r_c} ds e^{-pt(s)-\phi(s)}. \tag{3.50}$$

For $r_c \to r_h$, $\bar{D}_c^{EM}$ has the universal behavior

$$\bar{D}_c^{EM} = \frac{1}{4\pi} + \mathcal{O}(r_c - r_h) \tag{3.51}$$

for any geometry. Unsurprisingly, it follows from (3.10) that the leading term is exact for all r_c for Rindler space. For an AdS black p-brane with $p > 1$ and a constant gauge coupling ($\phi = 0$) the integral is easily evaluated and yields

$$\bar{D}_c^{EM} = \frac{p+1}{4\pi(p-1)} \frac{1 - (\frac{r_h}{r_c})^{p-1}}{1 - (\frac{r_h}{r_c})^{p+1}}. \tag{3.52}$$

Near the boundary $r_c = \infty$ we obtain

$$\bar{D}_\infty^{EM} = \frac{p+1}{4\pi(p-1)} + \mathcal{O}\left(\frac{r_h}{r_c}\right). \tag{3.53}$$

The leading term agrees with the result of Kovtun and Ritz [27] and Starinets [48]. We see that the dimensionless diffusivity runs from their result to the universal $\frac{1}{4\pi}$ in (3.51) as the cutoff is taken from infinity to the horizon.

3.3.4 Near horizon expansion

Here we consider the near-horizon dynamics when $r \to r_h$. There is then no need to make the long-wavelength ϵ expansion which required small ω and k. Inspired by the diffusion behavior of (3.73), or equivalently (3.49), we let $(\tau, r) \to (t, \rho)$ where

$$\tau = \frac{t}{\lambda}, \qquad r = r_h + \lambda\rho. \tag{3.54}$$

Taking the limit $\lambda \to 0$ corresponds to focusing on the near-horizon region. We also have

$$h(r) = \lambda\rho + \mathcal{O}(\lambda^2), \qquad e^{t(r)} = 1 + \mathcal{O}(\lambda). \tag{3.55}$$

Since we wish to consider couplings which are regular in r we can write

$$e^{\phi} = e^{\phi_0} + \mathcal{O}(\lambda) \tag{3.56}$$

where ϕ_0 is a constant. The gauge field has an associated expansion

$$F_{\mu\nu} = F^{(0)}_{\mu\nu} + F^{(1)}_{\mu\nu}\lambda + \mathcal{O}(\lambda^2). \tag{3.57}$$

At leading order (3.22) and (3.23) give

$$F^{(0)}_{i\rho} = 0, \qquad \partial_\rho F^{(0)}_{t\rho} = 0. \tag{3.58}$$

From (3.22) and (3.24) we find

$$\partial^i F^{(1)}_{i\rho} = \partial^2 F^{(0)}_{t\rho}, \qquad F^{(0)}_{ti} = (\rho - \rho_c)\, \partial_i F^{(0)}_{t\rho} + f_{ti}(t, x^j) \tag{3.59}$$

where f_{ti} is any function of (t, x^i). The cutoff surface is at $r_c = r_h + \lambda\rho_c$ and there we impose the boundary condition $F^{(0)}_{it} = 0$. Identifying $F^{(0)}_{t\rho} = e^{-\phi_0} q_c$ together with equation (3.21) at $\mathcal{O}(\lambda)$ leads to

$$\partial_t F^{(0)}_{t\rho} = \rho_c \partial^2 F^{(0)}_{t\rho} \tag{3.60}$$

which is equivalent to

$$\partial_t q_c = \rho_c \partial^2 q_c. \tag{3.61}$$

Alternatively we can write this in terms of the dispersion relation

$$i\omega_c = \frac{1}{4\pi T_c} k_c^2 \tag{3.62}$$

with *no subleading* k^4 *corrections for* $r_c \to r_h$. This makes sense since taking the cutoff to the horizon should correspond to an infrared limit in which higher derivative corrections are scaled away.

3.3.5 Subsummary

To summarize this section, for large r_c, which should correspond to taking the fluid cutoff to the UV, the charge density obeys a complicated non-linear dispersion relation. For small $\omega \sim k^2$, Fick's law holds (for any r_c) with a dimensionless diffusivity $\bar{D}^{EM}_c$ expressed as a radial line integral from the horizon to the cutoff. In the infrared limit as the cutoff is taken to the horizon, the dispersion relation simplifies to the Fick law and $\bar{D}^{EM}_c$ takes the universal value $\frac{1}{4\pi}$.

3.4 Gravity

Now we will adapt this to gravity. The linearized Einstein equations become rather complicated in the general background (5.14) but have been completely analyzed in a series of papers whose results we shall use without rederivation. One important feature, demonstrated in [25] and [19], is that the linearized gravity fluctuations decouple into so-called vector, scalar (or sound) and tensor type, characterized by their transformations under the $O(p)$ rotational symmetry. We consider these different types in turn.

We stress that throughout this section we assume that linearized metric fluctuations are governed by equations of motion whose form is given by the linearization of the Einstein tensor. R^2 corrections or even certain matter couplings could change this. Quite often these equations are unchanged, especially for the shear mode. We expect qualitative generalizations of our results to hold for any type of couplings.

3.4.1 Vector/shear modes

In this section we consider the vector fluctuations, which turn out to be the most interesting for our purposes. These have nonzero h_{ia} components which obey

$$\partial^i h_{i\tau} = 0, \quad h_{ij} = F(r)\partial_{(i}h_{j)\tau}. \tag{3.63}$$

In terms of these we define U(1) gauge fields

$$A^i = e^{-2t}h^i_a dx^a, \tag{3.64}$$

and field strengths

$$F^i = dA^i. \tag{3.65}$$

Here the index i labels the different field strengths. Following [22], by considering dimensional reduction along the polarization direction of the metric perturbations, the linearized Einstein equation for each F^i is precisely that of an abelian gauge field with position-dependent gauge coupling

$$e^{\phi} = e^{2t}. \tag{3.66}$$

This reduces the equation for the metric vector fluctuations to $p-1$ independent copies of the Maxwell equations. We may therefore read the answer off of the solution of the previous section. We take ingoing boundary conditions at the horizon and Dirichlet boundary conditions on $h^i_a(x^a, r_c)$ at the cutoff. This amounts to fixing the induced metric on Σ_c. The analog of the charge density q_c is the vector field

$$v^i(x^a) \equiv (\partial_r - 2t')h^i_\tau(x^a, r_c). \tag{3.67}$$

It follows from (3.63) that these are divergence free

$$\partial_i v^i = 0. \tag{3.68}$$

The analog of the Fick-Ohm law is the forced linearized Navier-Stokes equation

$$\partial_\tau v^i = D_c \partial^2 v^i + s^i \tag{3.69}$$

where the diffusivity is

$$D_c = \int_{r_h}^{r_c} ds e^{-(p+2)t(s)}. \tag{3.70}$$

Here the forcing term s_i is

$$s^i = e^{-2t(r_c)}(\partial_j \partial_\tau h^i_j(x^a, r_c) - \partial^2 h^i_\tau(x^a, r_c)). \tag{3.71}$$

We wish to define a dimensionless coordinate invariant diffusivity[11] $\bar{D}_c$ by transforming to proper coordinates and multiplying by the local temperature T_c. This proceeds exactly as in the electromagnetic case and the result is

$$\bar{D}_c = \frac{e^{2t(r_c)}h'(r_h)}{4\pi h(r_c)} \int_{r_h}^{r_c} dse^{-(p+2)t(s)}. \tag{3.72}$$

When the boundary conditions are chosen so that the forcing term vanishes, the transverse velocity fields v^i propagate with the dispersion relation[12]

$$i\omega_c = \bar{D}_c k_c^2. \tag{3.73}$$

The dimensionless diffusivity behaves universally in the infrared as the cutoff is taken to the horizon[13]

$$\bar{D}_c \to \frac{1}{4\pi} \text{ for } r_c \to r_h. \tag{3.74}$$

This agrees with the result obtained three decades ago in [5]. It applies for any geometry of the form (5.14). It is really a property of the linearized Einstein equation in Rindler space, which is the only relevant region for the calculation in the $r_c \to r_h$ limit.

[11] The quantity $\bar{D}_c$, which is the kinematic viscosity times the temperature, differs from the dynamic viscosity η by a factor involving the energy density, temperature and pressure. It is a more basic quantity from our perspective in that it is directly related to the measured decay rate of a perturbation. At this point we cannot define η, because we have not determined the energy density or pressure. This requires a bit more machinery - the Brown-York stress tensor - and will be worked out in section 5.

[12] We have set $\hbar = c = 1$, and the Newton constant G does not enter our calculations. In more general conventions $i\omega_c = \hbar c^2 \bar{D}_c k_c^2$. The $\hbar$ appears in this classical calculation because the diffusivity is expressed in terms of the local temperature which is itself proportional to $\hbar$.

[13] The results of [29–32, 34, 35] suggest the universal value will be modified by higher-derivative gravity corrections.

3.4.2 A D-theorem and other special properties of $\bar{D}_c$

So far the discussion of the electromagnetic and gravitational diffusivity have been exactly parallel. However some special features arise in the gravitational case. This first is that the integrand of (3.70) turns out to obey

$$e^{-(p+2)t} = \partial_r \left[\frac{he^{-(p+2)t}}{h' - 2t'h}\right] + 16\pi G \left[\frac{he^{-(p+2)t}}{(h' - 2t'h)^2}\right] T_{\mu\nu}\zeta^\mu\zeta^\nu. \tag{3.75}$$

Here $T_{\mu\nu}$ is the bulk matter stress tensor and ζ is any null vector tangent to the brane with time component $h^{-1/2}\partial_\tau$. Hence if there is no matter or if $T_{\mu\nu}\zeta^\mu\zeta^\nu = 0$, the integrand is a total derivative. $\bar{D}_c$ is then given by the simple expression

$$\bar{D}_c = \frac{h'(r_h)}{4\pi}\frac{e^{-pt(r_c)}}{h'(r_c) - 2t'(r_c)h(r_c)}. \tag{3.76}$$

The fact that the expression for the diffusivity can be integrated stems from the fact that the shear modes are pure gauge at zero momentum (we generalize here [48]). $h_{i\tau}$ is nonzero for $k^i = \omega = 0$, and we can derive its radial dependence from the zero momentum Einstein equation. However we can also solve this equation with a gauge transformation of the form

$$\delta x^i = e^i\tau + Be^i\int e^{-2t}dr, \qquad \delta\tau = -Be_i x^i \tag{3.77}$$

with B, e^i arbitrary constants. The second term in δx^i is added to preserve our gauge condition $h_{r\mu} = 0$. The nonzero component is

$$h_{i\tau} = \left(Bh + e^{2t}\right)e_i \tag{3.78}$$

In order to preserve the boundary condition $h_{i\tau}(r_c) = 0$, we must take

$$B = -\frac{e^{2t(r_c)}}{h(r_c)}, \tag{3.79}$$

resulting in

$$h_{i\tau}(r) = e_i\big(e^{2t(r)} - \frac{h(r)}{h(r_c)}e^{2t(r_c)}\big) \equiv f_1(r)e_i. \tag{3.80}$$

Of course we already found this solution, which is not pure gauge at non-zero-momentum, using the Einstein equation. The present derivation has the distinct advantage that it follows from gauge invariance and therefore is completely independent of the prescribed dynamics. Note that we can also find matter perturbations compatible with this gravity solution by the same method.

$\bar{D}_c$ can be expressed as a ratio of the coefficients of $\partial_r h_{i\tau}$ and $\partial_r h_{ij}$. The latter vanishes at zero momentum so we need to work to next order in ϵ expansion. We promote $e^i \to e^i(x^a)$, with $\partial_i e^i(x^a) = 0$. By symmetry $h_{ij} = e^{2t} f_2(r)(\partial_i e_j + \partial_j e_i)$ for some function $f_2(r)$, determined from the ij components of the linearized Einstein equation

$$\delta G_{ij} = 8\pi G \delta T_{ij}. \tag{3.81}$$

Again by symmetry the variation of T_{ij} with the matter fields is of the form $t_m(\partial_i e_j + \partial_j e_i)$ and (3.81) reduces to a single equation which becomes

$$e^{-(p-2)t}\partial_r[-e^{(p-2)t} f_1 + e^{tp} h \partial_r f_2] = -16\pi G t_m \tag{3.82}$$

in the long wavelength limit. In many cases (all that we have studied) $t_m = 0$ and hence

$$h(r)\partial_r f_2(r) = e^{-2t(r)} f_1(r) - f_1(r_h)e^{-pt(r)}. \tag{3.83}$$

This is then enough information to reproduce (3.76). So the universality of (3.76) is at least in part a consequence of zero-momentum gauge invariance.

Interestingly, if we plug in the neutral AdS black brane metric (3.11) we find the surprising result

$$\bar{D}_c = \frac{1}{4\pi} \tag{3.84}$$

for any value of r_c. Hence in this particular case $\bar{D}_c$ (unlike $\bar{D}_c^{EM}$ for the same geometry) does not run. In general one expects $\bar{D}_c$ and as well as all transport coefficients to run. Indeed for the charged black brane (3.14) we will exhibit a rather nontrivial RG flow in section 4.5 below. The constant value appearing in (3.84) is directly related to the fact that $\frac{\eta}{s}$ does not run, which will be discussed in section 6 below. However for now we briefly note the relation. The viscosity is related to $\bar{D}_c$ by the relation $\eta = \frac{\bar{D}_c(\mathcal{E}+\mathcal{P})}{T_c}$ where $\mathcal{E}$ ($\mathcal{P}$) is the energy density (pressure). In the absence of chemical potentials, the entropy density is determined from $T_c s = \mathcal{E} + \mathcal{P}$, implying $\bar{D}_c = \frac{\eta}{s}$. When there are chemical potentials such as for the charged black brane this simple formula no longer holds, and $\bar{D}_c$ runs.

We now show that, in a wide range of circumstances, $\bar{D}_c$ decreases with increasing r_c, i.e. it increases under RG flow into the infrared. We follow in spirit the A-theorem of [?]. Define the quantity

$$H(r_c) \equiv e^{(p-2)t(r_c)} h^2(r_c) \partial_{r_c}\Big(\frac{4\pi \bar{D}_c}{h'(r_h)}\Big) = e^{-2t} h - (h' - 2ht') e^{pt} \int_{r_h}^{r_c} ds e^{-(p+2)t(s)}. \tag{3.85}$$

Since $h(r_h) = 0$, we have $H(r_h) = 0$. The gradient of H then obeys

$$\partial_r H(r) = -16\pi G T_{\mu\nu} \zeta^\mu \zeta^\nu e^{pt(r)} \int_{r_h}^{r} ds e^{-(p+2)t(s)}. \tag{3.86}$$

This relation employs the Einstein equation

$$16\pi G T_{\mu\nu} \zeta^\mu \zeta^\nu = 2 G_{\mu\nu} \zeta^\mu \zeta^\nu = h'' + (p-2) h' t' - 2 t'' h - 2 p t'^2 h. \tag{3.87}$$

The null energy condition implies $T_{\mu\nu} \zeta^\mu \zeta^\nu \geq 0$, and hence that $\partial_r H \leq 0$. Since $H(r_h) = 0$ we conclude that $H(r) \leq 0$ for $r \geq r_h$. It then immediately follows from the definition (3.85) that

$$\partial_{r_c} \bar{D}_c \leq 0. \tag{3.88}$$

We expect that the general approach of this proof will be applicable to a wide range of situations. However, we wish to note that the precise result (3.88) may also be obtained from (3.76) simply by differentiating with respect to r_c.

3.4.3 Tensor modes

The tensor modes of the metric have nonzero components h_{ij} with $h_i{}^i = \partial_i h^i_j = 0$. The equations governing their fluctuations appear in [25], and are equivalent to a $(p+2)$-dimensional scalar Laplacian for each of the $(p-2)(p+1)/2$ tensor components. The analysis of these modes in our setup is very similar to the one given in section 3 for the F_{ij} modes of the electromagnetic fluctuations, and will not be spelled out here. The conclusion is that, in either the near-horizon or long-wavelength expansions (3.29), their values are everywhere fixed by the boundary conditions on Σ_c and the horizon. There are no dynamical modes in these expansions.

3.4.4 Scalar/sound modes

In this subsection we consider the scalar or sound mode. The equations of motion are somewhat complicated and we restrict ourselves here to the case of the AdS black threebrane so that $p = 3$, although we expect the more general case to be similar. For $p = 3$ the equations have been fully analyzed in [25]. They found that the metric fluctuations are determined by a certain linear combination of the components, denoted Z_2, obeying a second order radial differential equation which they express (equation 4.35) in Schwarzschild-like coordinates. For our purposes it is more convenient to work in the advanced coordinates (3.11), because these are smooth at the future horizon and the ingoing boundary condition is simply regularity at $r = r_h$. Z_2, which is a Fourier transform with respect to Schwarzschild time, is traded in these coordinates for the Fourier transform with respect to τ denoted here by X. X is related to Z_2 via a factor of $e^{i\omega \int^r \frac{dr'}{h}}$:

$$X = \left(\frac{r - r_h}{r + r_h}\right)^{\frac{-i\omega R^2}{4r_h}} Z_2 \tag{3.89}$$

Rewritten in terms of X and the coordinate r the equation for the sound mode, equation (4.35) of reference [25] , is

$$\begin{gathered}\left(r^4 - r_h^4\right)\partial_r^2 X + \left(r^4 - r_h^4\right)\left[\frac{5r^4 - r_h^4}{r\left(r^4 - r_h^4\right)} + \frac{8r_h^4 k^2}{r r_h^4 k^2 + 3r^5\left(-k^2 + \omega^2\right)} + \frac{iR^2\omega}{r^2 - r_h^2}\right]\partial_r X = \\ \left[k^2 R^4 - \frac{\omega^2 R^4\left(3r^2 + r_h^2\right)}{4(r^2 + r_h^2)} - \frac{i\omega R^2\left(3r^2 + r_h^2\right)}{2r} + \frac{4r_h^4 k^2\left(4r_h^4 - i\omega R^2 r\left(r_h^2 + r^2\right)\right)}{r^2(r_h^4 k^2 + 3r^4(-k^2 + \omega^2))}\right] X.\end{gathered} \tag{3.90}$$

Let us now analyze this equation in the long-wavelength expansion (3.29) with $\omega \sim k^2 \sim \epsilon^2 \to 0$. Note that there are no poles appearing in (3.90) for small $\omega \sim k^2$, so we can safely take $\epsilon \to 0$ for all r. One finds the equation for the leading term in the ϵ-expansion of X

$$\partial_r^2 X^0 + \left(\frac{5r^4 - r_h^4}{r(r^4 - r_h^4)} - \frac{8r_h^4}{r(3r^4 - r_h^4)}\right)\partial_r X^0 + \frac{16r_h^8}{r^2(3r^4 - r_h^4)(r^4 - r_h^4)} X^0 = 0, \tag{3.91}$$

which does not depend on ω or k. This has a unique solution, up to an overall scale, which is non-singular at $r = r_h$. The scale is then fixed by the boundary condition at the cutoff r_c. Unsurprisingly we learn that there are no dynamical degrees of freedom in the sound mode in the expansion (3.29).

It is interesting, but outside the scope of this paper, to consider an alternate scaling limit in which $\omega \sim k$. In this limit nontrivial sound modes may appear with some fixed velocity v_s. In the limit (3.29), all fixed velocities including v_s are sent to infinity.

Now let us now consider the near horizon expansion. In this expansion we take $r_c \to r_h$ without any preassumed relation between ω and k. In this case it is not so simple to take

$r_c \to r_h$, as a quick inspection of (3.90) indicates there may be poles at

$$\omega = \pm\sqrt{\frac{2}{3}}k. \tag{3.92}$$

Let us first consider the case where ω does not take the value (3.93). Then we can safely take $r_c \to r_h$, and as in our long-wavelength expansion above there are no degrees of freedom. Hence the only possibility for dynamical modes are those that obey (3.93). Equation (3.93) can be written in terms of the coordinate-invariant proper quantities and local temperature as

$$\omega_c = \pm\sqrt{\frac{2}{3}}\frac{e^{t(r_c)}T_c}{T_H}k_c. \tag{3.93}$$

This is a dispersion relation for a sound mode with velocity of sound

$$v_s = \sqrt{\frac{2}{3}}\frac{e^{t(r_c)}T_c}{T_H}. \tag{3.94}$$

However note that as $r_c \to r_h$, T_c and hence the velocity of sound goes to infinity. Hence, in an expansion in $r_c - r_h$, no sound modes appear and the fluid at the horizon is incompressible.

This again is consistent with the expectation that the limit $r_c \to r_h$ is a nonrelativistic, low energy limit. It is low energy because of the high redshifts, and non-relativistic because the degeneracy of the induced metric on Σ_c appears only in the temporal and not the spatial components. In its most general form, the Navier-Stokes equation for a fluid contains both sound and shear modes. However one may take a further limit of these equations of the form (3.29) in which velocities scale as $v^i \to \epsilon v^i \to 0$ (see [42] for a nice discussion). In this limit the sound velocity goes to infinity. The fluid retains its nonlinear interactions and is described by the incompressible Navier-Stokes equation. The near-horizon limit resembles a bulk version of this limit.

This leads us to the interesting conclusion that the fluid which lives at the horizon is, at linear order, universally given by an incompressible fluid with dimensionless diffusivity $\bar{D}_c = \frac{1}{4\pi}$.

3.4.5 Charged black brane

In this section we consider the case of the charged black brane geometry (3.14) as a somewhat more non-trivial illustration of our approach. If symmetry allows, a metric perturbation can source a matter perturbation already at the linear level. For the charged black brane there is a matter perturbation of the general form

$$\delta A = a_j dx^j. \tag{3.95}$$

We can solve the coupled equations by transforming them back to the previous case. First, let us define a shift of the "gravitational" field strength appearing in (3.64) and (3.65) by

$$\tilde{F}^j_{\tau r} = F^j_{\tau r} - 16\pi G a^j e^{-2t} A'_\tau(r), \tag{3.96}$$

with A_τ given in equation (3.14). (Note that, in a notational clash, *neither* F nor $\tilde{F}$ here is the field strength of A! Instead, F refers to the "gravitational" field strength as defined in (3.64) and (3.65).) $\tilde{F}^j_{\tau r}$ then obeys exactly the same equations (3.21) through (3.23) obeyed by $F^j_{\tau r}$ in the neutral case.

Consequently, following the logic in Section 4.3, we find at lowest order, similarly to (3.34),

$$\tilde{F}^j_{\tau r} = -e^{-(p+2)t}\tilde{v}^j(x^a) \tag{3.97}$$

where

$$\tilde{v}^j(x^a) \equiv \tilde{F}^j_{r\tau}(x^a, r_h). \tag{3.98}$$

Note that the effective "charge" $\tilde{v}^j$ in (3.98) is modified from the corresponding equation (3.27).

Now we must examine the two remaining equations: the lowest order Bianchi identity

$$\partial_r F^j_{i\tau} = -\partial_i F^j_{\tau r} \tag{3.99}$$

and the Maxwell equation for the gauge perturbation a^j

$$F^j_{\tau r} = \frac{1}{Q}\partial_r\left(e^{(p-2)t} h \partial_r a^j\right). \tag{3.100}$$

Note that we have written these two equations in terms of $F^j_{\tau r}$; by doing so we can now quickly see that the right hand side of (3.99) is just a total derivative. Thus we find, similarly to (3.59) for $r = r_h$,

$$F^j_{i\tau}\left(x^a, r_h\right) = \frac{1}{Q}\int_{r_h}^{r_c} dr\ \partial_r\left(e^{(p-2)t} h \partial_i \partial_r a^j\right) + f_{i\tau}(x^a). \tag{3.101}$$

Before we can use (3.101) to find the diffusivity D_c, we must first find a^j. To do so, we simply plug (3.97) and (3.100) into (3.96), additionally imposing the boundary condition $a^j(r_c) = 0$. We thus find

$$a^j(r) = \frac{\tilde{v}^j(x^a)}{16\pi G Q}\left[1 - e^{pt(r)-pt(r_c)}\frac{h'(r) - 2t'(r)h(r)}{h'(r_c) - 2t'(r_c)h(r_c)}\right]. \tag{3.102}$$

Again we can find a dimensionless coordinate invariant diffusivity which is

$$\begin{aligned}\bar{D}^Q_c &= \frac{h'(r_h)}{4\pi}\frac{e^{-pt(r_c)}}{h'(r_c) - 2t'(r_c)h(r_c)} \\ &= \frac{p+1-\alpha Q^2\left(p-1\right)}{4\pi\left[p+1+\alpha Q^2\left(p+1-2p\frac{r_h^{p-1}}{r_c^{p-1}}\right)\right]}.\end{aligned} \tag{3.103}$$

Note that the first line of (3.103) is identical to (3.76), corroborating the universality of (3.76). However when we plug in the metric coefficients for the charged black brane to get the second line of (3.103), we see that, unlike the case of the neutral black brane, the diffusivity is no longer constant.

3.4.6 Asymptotically flat black p-brane

In our next example we consider the asymptotically flat S^3-compactified black NS5 solution (3.15). The 7-dimensional effective action for the rotationally invariant modes is Einstein gravity plus scalars, hence according to section 4.2, $\partial_{r_c}\bar{D}_c = 0$, and $\bar{D}_c = \frac{\eta}{s} = \frac{1}{4\pi}$ at any scale. Of course this can be reproduced by direct calculation. The local temperature is a non-trivial function of radial position:

$$T_c = \frac{1}{2\pi y_h y_c^{3/5}} \left(1 + \frac{L^2}{y_h^2}\right)^{-1/2} \left(1 + \frac{L^2}{y_c^2}\right)^{-1/10} \left(1 - \frac{y_h^2}{y_c^2}\right)^{-1/2}. \tag{3.104}$$

General formulae from the following section for the energy density plus pressure give

$$\mathcal{E} + \mathcal{P} = \frac{y_h^2}{8\pi G y_c^{18/5}} \left(1 + \frac{L^2}{y_c^2}\right)^{-3/5} \left(1 - \frac{y_h^2}{y_c^2}\right)^{-1/2} \tag{3.105}$$

and entropy density

$$s = \frac{e^{-5t(r_c)}}{4G} = \frac{y_h^3}{4G y_c^3} \left(1 + \frac{L^2}{y_c^2}\right)^{-1/2} \left(1 + \frac{L^2}{y_h^2}\right)^{1/2}. \tag{3.106}$$

We note that $\mathcal{E} + \mathcal{P} = T_c s$. The $L \to \infty$ throat limit of these expressions should describe the thermodynamics of the quantum theory on the NS5-brane, but we will not further pursue this here.

3.4.7 Background stress-energy

Now we evaluate t_{ab} for our general metric (5.14) at the cutoff hypersurface Σ_c. The unit normal is

$$n = \sqrt{h}\partial_r + \frac{1}{\sqrt{h}}\partial_\tau \tag{3.107}$$

and

$$K_{\mu\nu}dx^\mu dx^\nu = \sqrt{h}\left[-\frac{h'}{2}\left(d\tau - \frac{dr}{h}\right)^2 + t' e^{2t} dx_i dx^i\right], \tag{3.108}$$

$$\gamma_{\mu\nu}dx^\mu dx^\nu = \left[-h\left(d\tau - \frac{dr}{h}\right)^2 + e^{2t}dx_i dx^i\right]. \tag{3.109}$$

In intrinsic coordinates to Σ_c as defined in (3.6)

$$K_{ab}dx^a dx^b = -\frac{h'}{2\sqrt{h}}(dx_c^0)^2 + \sqrt{h}t' dx_{ci}dx_c^i, \tag{3.110}$$

$$\gamma_{ab}dx^a dx^b = \eta_{ab}dx_c^a dx_c^b. \tag{3.111}$$

The leading order Brown-York stress tensor is

$$t^0_{ab}dx^a dx^b = \frac{\sqrt{h}}{8\pi G}\big(-pt'(dx_c^0)^2 + \big((p-1)t' + \frac{h'}{2h}\big)dx_{ic}dx_c^i\big) + C'\eta_{ab}dx_c^a dx_c^b, \tag{3.112}$$

where all r-dependent quantities are evaluated at $r = r_c$. This is the stress tensor of a fluid at rest with constant pressure $\mathcal{P}$ and energy density $\mathcal{E}$. The constant part of the difference $\mathcal{E} - \mathcal{P}$ depends on the choice of constant C: the behavior of the linearized fluid depends only on the sum

$$\mathcal{E} + \mathcal{P} = \frac{\sqrt{h}}{8\pi G}\left(\frac{h'}{2h} - t'\right). \tag{3.113}$$

Note that for $r_c \to r_h$, $\mathcal{E} + \mathcal{P} \to \frac{T_c}{4G} = T_c s$, where s is the entropy density of the horizon.

3.4.8 Perturbations

Now we consider a small perturbation h. It is convenient to choose the gauge

$$h_{rr} = h_{r\tau} = h_{ri} = 0, \tag{3.114}$$

so that the metric retains the general form (5.14). Moreover, our boundary condition implies

$$h_{ab}(r_c) = 0. \tag{3.115}$$

It then follows that the leading correction to the extrinsic curvature in the long wavelength limit simplifies to $2K^1_{ab} = \sqrt{h}\partial_r h_{ab}$, and

$$t^1_{ab}(x^a, r_c) = \frac{\sqrt{h}}{16\pi G}\big(-\partial_r h_{ab} + \gamma_{ab}\gamma^{cd}\partial_r h_{cd}\big). \tag{3.116}$$

For the shear mode, only $h_{i\tau}$ and h_{ij} are nonzero. Moreover, as shown in [**?**] the Einstein equation implies they are related by $F\partial_{(j}h_{i)\tau} = h_{ij}$ for some function $F(r)$, as in equation (3.63) previously. Conservation of the Brown-York stress tensor implies that $F(r_c) = \frac{\bar{D}_c}{T_H}$ on Σ_c and

$$\partial_r h_{ij} = \frac{\bar{D}_c}{T_H}\partial_r\partial_{(i}h_{j)\tau}. \tag{3.117}$$

Defining the velocity field

$$v_i = -\frac{e^{-t}}{16\pi G(\mathcal{E}+\mathcal{P})}\partial_r h_{i\tau}, \tag{3.118}$$

The linearized stress tensor can be written in intrinsic coordinates

$$t^1_{ab}dx^a_c dx^b_c = 2(\mathcal{E}+\mathcal{P})v_i dx^i_c d\tau_c + \eta\partial_i v_j dx^i_c dx^j_c \tag{3.119}$$

with viscosity given by

$$\eta = \frac{\bar{D}_c(\mathcal{E}+\mathcal{P})}{T_c}. \tag{3.120}$$

Using (3.48, 3.76) and (3.113), this becomes

$$\eta(r_c) = \frac{e^{-pt(r_c)}}{16\pi G}, \tag{3.121}$$

in agreement with [5] for $r_c \to r_h$ where $t \to 0$.

We see from the above that the stress tensor for the linearized shear mode indeed takes the form of an incompressible fluid.

3.5 RG flow, the first law of thermodynamics and $\frac{\eta}{s}$ universality

In this section we want to show that the first law of thermodynamics together with isentropy of the radial flows is equivalent to a radial component of the Einstein equation, and moreover imply that $\frac{\eta}{s}$ does not run. A general type of equivalence between the Einstein equation and the first law has been demonstrated by Jacobson [54], see also [55]. We suspect our equivalence is related to this - as well as the recent work [56] - but we defer this issue to future consideration.

For the present purposes, it is convenient to consider a quotient of the general geometry (5.14) under shifts of x^i

$$x^i \sim x^i + n^i, \qquad n^i \in Z. \tag{3.122}$$

This turns the spatial R^p in the p-torus T^p with r-dependent volume

$$V_p = e^{pt}. \tag{3.123}$$

The total entropy $S = sV_p$ as a function of the total energy $E = \mathcal{E}V_p$, pressure $\mathcal{P}$, charge Q, chemical potential μ, inverse temperature $\beta = T_c^{-1}$ and volume V_p are related by

$$S = \beta E + \beta \mathcal{P} V_p - \beta \mu Q. \tag{3.124}$$

In general there will be one μQ term for each chemical potential present; in our case we consider only the charged brane given in (3.14), for which $\mu = A_\tau/\sqrt{h}$.

Now let us consider the first law of thermodynamics for radial variations under the assumption that the variation is isentropic. This reads

$$0 = \partial_r S = \partial_r \beta \left(E + \mathcal{P}V_p - \mu Q\right) + \beta \left(\partial_r E + \partial_r \mathcal{P} V_p + \mathcal{P} \partial_r V_p - \partial_r \mu Q\right). \tag{3.125}$$

Using expressions (3.48, 3.113) and (3.123) for the thermodynamic quantities[14] one finds (3.125) becomes

$$\partial_r S = \frac{V_p}{16\pi G T_H}(h'' + (p-2)h't' - 2t''h - 2pt'^2 h) - \frac{V_p}{T_H} T_{\mu\nu}\zeta^\mu\zeta^\nu. \tag{3.126}$$

Here we have also used $V_p T_{\mu\nu}\zeta^\mu\zeta^\nu = Q^2 e^{-pt}$ for the charged brane. Comparing with (3.87), we see that the right hand side is exactly a component of the Einstein equations. Therefore, isentropy of the RG flow implies a radial Einstein equation. Of course this can be turned around to state that the radial Einstein equation implies the isentropy of RG flow.

Now if $S = constant$, then

$$s = \frac{e^{-pt}}{4G}, \tag{3.127}$$

where the overall multiplicative factor is fixed by demanding the Bekenstein-Hawking law $s = \frac{1}{4G}$ at the horizon. Combining with (3.121), we deduce that on Σ_c

$$\frac{\eta(r_c)}{s(r_c)} = \frac{1}{4\pi} \tag{3.128}$$

for any value of r_c.[15] We have already seen in our formalism that (3.128) for the special value $r_c = r_h$ is a universal feature of Rindler space. Now we see that under quite general assumptions, it does not change under RG flow, and so the value (3.128) will also apply to $r_c = \infty$, in agreement with findings in [26, 31]. So far we have not asked the question: *why* should radial evolution be isentropic? A physical answer from the gravity side is that the only entropy associated with a classical solution is the horizon of the black hole. Therefore the total entropy inside any hypersurface outside the horizon should be independent of radius.

[14] We are considering here only the thermodynamic quantities associated to the background. It would be interesting, but beyond the scope of this paper, to understand if this reasoning applies at quadratic order in the fluctuations, where shear dissipation produces entropy.

[15] Closely related observations were made in a different formalism by Iqbal and Liu [26].

This also gives a clue as to when the relation (3.128) might be violated [57]. When quantum corrections are included on the gravity side, the entropy will generically depend on radius. For example there might be a thermal gas of Hawking radiation surrounding the black hole or entanglement entropy across Σ_c. These are suppressed by a factor of $\hbar$ relative to the horizon entropy. We then see no reason to expect that (3.128) should survive such quantum corrections. The universality of (3.128) is presumably a statement about the classical gravity limit. We note of course that the classical gravity limit is in general not the same as the classical limit of the quantum theory underlying the fluid.

Chapter 4

From Navier - Stokes to Einstein

Our basic construction is roughly as follows. We begin with the region of $p+2$-dimensional Minkowski space inside a hypersurface Σ_c given by an equation of the form $x^2 - t^2 = 4r_c$. Σ_c is intrinsically flat (being the translation of an hyperbola in the t-x plane along the remaining p spatial directions), but has an extrinsic curvature linked to the constant acceleration $a = 1/\sqrt{4r_c}$. It asymptotes to its future horizon $\mathcal{H}^+$ which is the null surface $x = t$. We then study the effect of finite perturbations of the extrinsic curvature of Σ_c while keeping the intrinsic metric flat. These generically lead, when evolved radially inward with the Einstein equation, to singularities on $\mathcal{H}^+$. The special ones which are smooth on $\mathcal{H}^+$ are analyzed in the hydrodynamic "ϵ-expansion", which is a nonrelativistic, long-wavelength expansion and, importantly, keeps terms that are nonlinear in the size of the perturbation. It is found that tensor and scalar modes of the metric decouple in this limit and the remaining degrees of freedom are vector modes governed by the Navier-Stokes equation in $p + 1$ dimensions. We present (equation (4.12) below) the $p + 2$-dimensional solution of the Einstein equation through third order in the hydrodynamic expansion parameter ϵ. The first term is flat space. The second and third terms are algebraically constructed from the velocity field v^i and pressure P of an incompressible fluid. The nonlinear spacetime Einstein equation then reduces to the nonlinear incompressible Navier-Stokes equation for the pair (v^i, P).

This result is already interesting and non-trivial, but the fact that the Navier-Stokes arises when the geometric variables are subject to the same kind of expansion used in fluid dynamics might have been anticipated. A deeper connection appears when we consider an alternate expansion in which, instead of going to long distances, we take the acceleration of Σ_c to infinity. This is a near-horizon limit since it pushes Σ_c towards its horizon $\mathcal{H}^+$. We then show that, after a constant overall rescaling of the metric, *the near-horizon expansion is mathematically identical to the hydrodynamic expansion*. Hence the solutions of the Einstein equation (constrained by the boundary conditions of a flat metric on Σ_c and smoothness on $\mathcal{H}^+$) in this near-horizon expansion are in one-to-one correspondence with solutions of the incompressible Navier-Stokes equation. This then is the precise mathematical sense in which horizons are incompressible fluids.

It is possible that the ultimate origin of this relation is a deep and exact holographic duality relating (among other things) quantum black holes to fluids as has been suggested by string theoretic investigations. However in this paper we have concentrated on simply establishing the mathematical relationship between (1.12) and (1.11) in a manner which makes no assumptions about or reference to this tantalizing possibility.

This chapter organized as follows. Section 2 briefly reviews the hydrodynamic expansion in the study of fluids, and the emergence of the incompressible Navier-Stokes equation in the hydrodynamic limit. In section 3 we specify the boundary conditions, explained roughly above, used to isolate horizon dynamics. In section 4 we present the of the nonlinear Einstein equation with these boundary conditions through the first three orders in the hydrodynamic expansion, and show that the first nontrivial term corresponds to the velocity field of an incompressible fluid. We also discuss the geometric analog of forcing the fluid, argue for uniqueness, and discuss the possible formation of black hole type singularities. Section 5 presents a simpler form of the metric and shows that, up to an overall rescaling and after an appropriate coordinate transformation, it depends only on the product of the leading-order acceleration of Σ_c and the hydrodynamic expansion parameter ϵ. In section 6 we show that the geometries are, through the order constructed, of a special type known in four dimensions as Petrov type II. This may enable a connection of the present work with the large literature on algebraically special spacetimes [58–60]. Finally in section 8 we demonstrate, using the simplified metric of section 6, the equivalence of the hydrodynamic and near-horizon expansions.

4.1 The hydrodynamic limit and the ϵ-expansion

The incompressible Navier-Stokes equation has a well-known scaling symmetry which is important in the following and briefly reviewed here. Let the pair (v_i, P) solve the incompressible Navier-Stokes equation

$$\partial^i v_i = 0, \qquad \partial_\tau v_i - \eta \partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0, \tag{4.1}$$

where η is the kinematic viscosity and $i = 1, ...p$. Now consider a family of pairs $(v_i^\epsilon, P^\epsilon)$ in which frequencies and wavelengths are non-relativistically dilated and amplitudes scaled down by the parameter ϵ:

$$v_i^\epsilon(x^i,\tau) = \epsilon v_i(\epsilon x^i, \epsilon^2\tau), \tag{4.2}$$

$$P_i^\epsilon(x^i,\tau) = \epsilon^2 P(\epsilon x^i, \epsilon^2\tau).$$

It is easy to check that (4.1) directly implies

$$\partial_\tau v_i^\epsilon - \eta\partial^2 v_i^\epsilon + \partial_i P^\epsilon + v^{\epsilon j}\partial_j v_i^\epsilon = 0. \tag{4.3}$$

Hence (4.2) generates from the original solution a family of solutions parameterized by ϵ.

In real fluids there are always corrections to the Navier-Stokes equation. Galilean invariance requires that these vanish for constant v_i. Typical corrections are for example of the form

$$\partial_\tau v_i - \eta\partial^2 v_i + \partial_i P + v^j\partial_j v_i + v^k v^j \partial_k\partial_j v_i + \partial_\tau^2 v_i = 0. \tag{4.4}$$

If (v_i, P) obey this equation, the rescaled quantities obey

$$\partial_\tau v_i^\epsilon - \eta\partial^2 v_i^\epsilon + \partial_i P^\epsilon + v^{\epsilon j}\partial_j v_i^\epsilon + \epsilon^2\left(v^{\epsilon k} v^{\epsilon j}\partial_k\partial_j v_i^\epsilon + \partial_\tau^2 v_i^\epsilon\right) = 0. \tag{4.5}$$

The limit $\epsilon \to 0$ is the hydrodynamic limit. In this limit these corrections become irrelevant. Similarly the speed of sound goes to infinity and compressible fluids become incompressible. It is not hard to show that all reasonable types of corrections are scaled away, and the incompressible Navier-Stokes equation universally governs the hydrodynamic limit of essentially any fluid. The limit is an incredibly rich and interesting one because, even though the amplitudes are scaled to zero, nonlinearities survive. It is this hydrodynamic limit of a fluid that we will match to a near-horizon limit in gravity.

4.2 Characterizing the dual geometries

We seek a relation between the (p+2)-dimensional Einstein and (p+1)-dimensional Navier-Stokes equations. Of course, the former has a much larger solution space than the latter so only a special type of Einstein geometry is relevant. Roughly speaking, the relevant geometries are non-singular perturbations of a horizon. Let us now make this more precise.

We consider geometries of the type depicted in Figure 4.1 with an outer "cutoff" boundary denoted Σ_c. The boundary hypersurface Σ_c is taken to be asymptotically null in both the far future and far past. In the Minkowskian coordinates $ds^2_{p+2} = -dudv + dx_i dx^i$, past null infinity $\mathcal{I}^-$ is the union of the null surfaces $v \to -\infty$ together with $u \to -\infty$ and Σ_c is the timelike hypersurface $uv = -4r_c$ with $v > 0$. Past (future) event horizons $\mathcal{H}^-$ ($\mathcal{H}^+$) are defined by the boundaries of the causal future (past) of Σ_c.

The dual geometries will be constructed in two a priori different expansions about Minkowski space: the near-horizon and the hydrodynamic ϵ-expansion. Ultimately the two expansions will be shown to be equivalent.

Initial data can be specified on the union of Σ_c and $\mathcal{I}^-$. We consider initial data which is asymptotically Minkowskian and flat (no incoming waves) on $\mathcal{I}^-$ (or equivalently $\mathcal{H}^-$). On Σ_c we generally demand that the intrinsic metric γ_{ab} be flat,

$$\gamma_{ab} = \eta_{ab}, \quad a, b = 0, ...p \tag{4.6}$$

although we will later consider "forcing" the system by perturbing γ_{ab}.

We wish to consider the general solution of the Einstein equations consistent with this initial data and smooth on $\mathcal{H}^+$.[1] In particular, so far we have not specified the extrinsic

[1] Here we allow for incoming flux where $\mathcal{I}^-$ meets Σ_c at $u = -\infty,\ v = 0$.

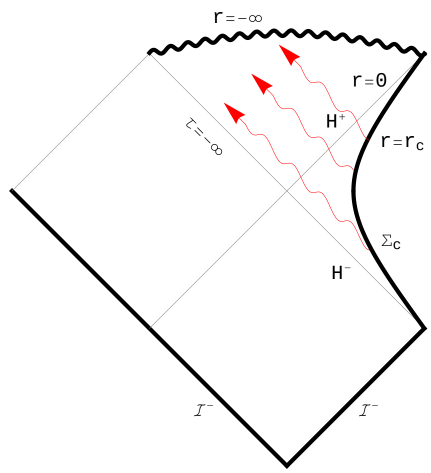

Figure 4.1: This figure depicts the Einstein geometry holographically dual to a fluid. The accelerated boundary hypersurface Σ_c at radius $r = r_c$ is intrinsically flat but the extrinsic curvature is given by the fluid stress tensor. This extrinsic curvature leads to gravity waves which propagate radially inward. The leading-order condition that these waves do not cross the past horizon $\mathcal{H}^-$ of Σ_c at $\tau = -\infty$ or produce singularities on the future horizon $\mathcal{H}^+$ at $r = 0$ is the non-linear incompressible Navier-Stokes equation for the fluid.

curvature K_{ab} on Σ_c or equivalently (and more conveniently) the Brown-York stress tensor on Σ_c[2]

$$T_{ab} \equiv 2(\gamma_{ab}K - K_{ab}). \tag{4.7}$$

If no initial data were prescribed on $\mathcal{I}^-$, any T_{ab} on Σ_c consistent with the constraint equations could be chosen. This data could then in general be evolved radially inwards to produce a spacetime everywhere inside of Σ_c. In general, such a spacetime will have gravitational flux (if not singularities) going up to $v = \infty$ ($\mathcal{I}^+$) as well as down to $\mathcal{I}^-$. Hence we have a "shooting problem" to find those special allowed choices of T_{ab} which produce a spacetime

[2]Our normalization here agrees with the conventional one for $G = 1/16\pi$.

smooth on $\mathcal{H}^+$ with no flux coming up from $\mathcal{I}^-$.

We solved this problem in [37] to leading order in a double expansion in long wavelengths and weak fields.[3] Ingoing Rindler coordinates were used for which the leading order flat metric is

$$ds_{p+2}^2 = -r d\tau^2 + 2d\tau dr + dx_i dx^i. \tag{4.8}$$

Σ_c is the accelerated surface $r = r_c$, $\mathcal{H}^-$ is $\tau = -\infty$ and $\mathcal{H}^+$ is $r = 0$. These coordinates are convenient for analyzing smoothness on $\mathcal{H}^+$. It was found that the allowed choices of T_{ab} are precisely those corresponding to the linearized fluid:

$$r_c^{3/2} T^{\tau i} = v^i, \quad r_c^{3/2} T^{ij} = -\eta \partial^{(i} v^{j)}, \tag{4.9}$$

where the (kinematic) viscosity here is given by the formula

$$\eta = r_c, \tag{4.10}$$

while v_i obeys the linearized incompressible Navier-Stokes equation

$$\partial_i v^i = 0, \quad \partial_\tau v^i - \eta \partial^2 v^i = 0. \tag{4.11}$$

If we choose any value for the viscosity other than (4.10), the constraint equations on Σ_c are still obeyed, but gravitational waves are propagated down to $\mathcal{I}^-$ and there is a singularity at $r = 0$.

In this paper we go one step further and solve the problem in certain hydrodynamic and near-horizon limits *without* making a linearized approximation, enabling us to see a direct connection between the nonlinear structures of the Navier-Stokes and Einstein equations.

[3] Our conventions here differ from [37].

4.3 Nonlinear solution in the ϵ-expansion

In this section we will improve on the analysis of [37] by solving the shooting problem in the long wavelength ϵ-expansion without a simultaneous linearized expansion. The general solution will be parameterized by a solution $v_i(x^i, \tau)$, $P(x^i, \tau)$ of the full nonlinear Navier-Stokes equation with viscosity (4.10) together with the parameter ϵ.

4.3.1 The solution

Consider the metric

$$\begin{aligned} ds^2_{p+2} =& - r d\tau^2 + 2 d\tau dr + dx_i dx^i \\ & - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\ & + \left(1 - \frac{r}{r_c}\right)\left[(v^2 + 2P) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\ & - \frac{(r^2 - r_c^2)}{r_c} \partial^2 v_i dx^i d\tau + \ \dots \end{aligned} \tag{4.12}$$

where $v_i = v_i(x^i, \tau)$ and $P(x^i, \tau)$ are independent of r. Here and henceforth $i, j = 1, ..p$ indices are raised and lowered with δ_{ij} and we take

$$v_i \sim \mathcal{O}(\epsilon), \qquad P \sim \mathcal{O}(\epsilon^2), \qquad \partial_i \sim \mathcal{O}(\epsilon), \qquad \partial_\tau \sim \mathcal{O}(\epsilon^2) \tag{4.13}$$

as in the hydrodynamic scaling of section 3. It follows that the first line on the right hand side of (4.12) is $\mathcal{O}(\epsilon^0)$ and each subsequent line is one higher order in ϵ. The linearization of this expression in v_i agrees with the linearized solution studied in [37].

On the cutoff surface Σ_c, $r = r_c$ and the induced metric is flat:

$$\gamma_{ab} dx^a dx^b = -r_c d\tau^2 + dx_i dx^i, \tag{4.14}$$

and hence satisfies the desired boundary condition. Here and henceforth $x^a \sim (x^i, \tau)$. The extrinsic curvature and unit normal on Σ_c are

$$\begin{aligned} K_{ab} &= \tfrac{1}{2}\mathcal{L}_N \gamma_{ab} = -\tfrac{1}{2}\left[T_{ab} - \tfrac{1}{p}\gamma_{ab}\gamma^{cd}T_{cd}\right], \\ N^\mu \partial_\mu &= \tfrac{1}{\sqrt{r_c}}\partial_\tau + \sqrt{r_c}\left(1 - \tfrac{P}{r_c}\right)\partial_r + \tfrac{v^i}{\sqrt{r_c}}\partial_i + \mathcal{O}(\epsilon^3). \end{aligned} \tag{4.15}$$

The Brown-York stress tensor is

$$T_{ab}dx^a dx^b = \frac{dx_i^2}{\sqrt{r_c}} + \frac{v^2}{\sqrt{r_c}}d\tau^2 - 2\frac{v_i}{\sqrt{r_c}}dx^i d\tau + \frac{(v_i v_j + P\delta_{ij})}{r_c^{3/2}}dx^i dx^j - 2\frac{\partial_i v_j}{\sqrt{r_c}}dx^i dx^j + \mathcal{O}(\epsilon^3). \tag{4.16}$$

We wish to solve the Einstein equations as a power series in ϵ. We first consider the necessary but not sufficient condition that the constraints be satisfied on Σ_c. At order ϵ^0 the metric is flat and T_{ab} is constant so they are trivially satisfied. The only way to get an order ϵ term is with one power of v^i and no derivatives. Such a linear term cannot appear because the constant v^i terms in (4.12) can, through quadratic order, be obtained from a boost of flat space. The first nontrivial equation is encountered at order ϵ^2:

$$r_c^{3/2}\partial_a T^{a\tau} = \partial_i v^i = 0. \tag{4.17}$$

This equation is satisfied if and only if v^i is the velocity field of an incompressible fluid. Taking this to be the case, one finds at order ϵ^3:

$$r_c^{3/2}\partial^a T_{ai} = \partial_\tau v_i - \eta\partial^2 v_i + \partial_i P + v^j \partial_j v_i = 0. \tag{4.18}$$

This is satisfied if and only if v^i solves the Navier-Stokes equation with pressure P and viscosity $\eta = r_c$.

Once the constraints are satisfied it is ordinarily possible to evolve the solution off the hypersurface, in this case in the radial direction, at least for a finite distance. Here we have the danger of singularities at the horizon $\mathcal{H}^+$ near $r = 0$, or equivalently waves coming up from

$\mathcal{I}^-$. We know from [37] that such singularities are absent in the linearized analysis provided the fluid viscosity takes the required value (4.10). We have checked by direct computation that this absence of singularities extends to the nonlinear case as well. That is all components obey

$$G_{ra}, G_{ab}, G_{rr} = \mathcal{O}(\epsilon^4) \tag{4.19}$$

and are nonsingular for finite values of r. Presumably the order ϵ^4 and higher terms in the metric can be chosen so that the Einstein equations are solved exactly.

It turns out that it is still possible to solve the Einstein equations analytically through order ϵ^3 with the "wrong" value of the viscosity (i.e. $\eta \neq r_c$) even in the nonlinear case. As expected these solutions develop a singularity at $r = 0$ near $\mathcal{H}^+$, and are presented in Appendix A .

4.3.2 Forcing the fluid

Solutions of the linearized Navier-Stokes equation decay exponentially in the future. There is some expectation - although no proof - that nonlinear solutions eventually decay as well. Therefore the extrinsic curvature on Σ_c in our examples is expected to become constant.

On the other hand, already at the linear level, Navier-Stokes solutions grow exponentially in the far past and typically are singular at $\tau = -\infty$. Therefore we expect that the dual geometry is also singular at $\tau = -\infty$, which is the past horizon $\mathcal{H}^-$ of Σ_c. This singularity is not problematic for real fluids, as we are typically interested in cases where forcing terms correct the Navier-Stokes equation. For example we might consider a fluid which is initially at rest, stirred at time $\tau = \tau_*$, and then left to evolve according to the unforced Navier-Stokes equation.

In fact this kind of situation is also very natural to consider on the gravity side. Consider flat Minkowski spacetime with a flat metric and constant extrinsic curvature on the boundary Σ_c for $\tau < \tau_*$. We then stir it at $\tau = \tau_*$ by momentarily perturbing the boundary condition that the induced metric on Σ_c be flat. This will send out a gravitational shock wave along $\tau = \tau_*$ and excite the geometry for $\tau > \tau_*$. The result should be an appropriate gluing of (4.12) along a null hypersurface to flat Minkowski space. This is depicted in Figure 4.2.

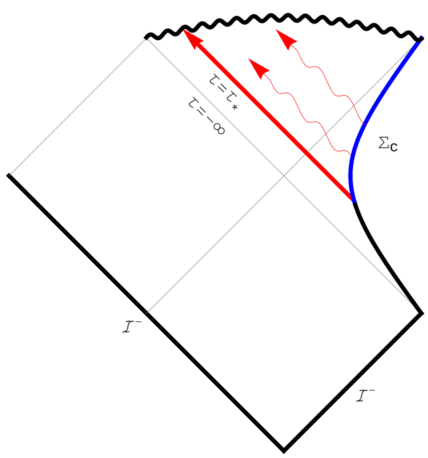

Figure 4.2: On the Σ_c surface, prior to $\tau = \tau_*$, all initial data is trivial. At $\tau = \tau_*$, a gravitational shock wave arrives. The shock forces the fluid on Σ_c, and consequently the v_i is nontrivial on Σ_c after τ_*.

At the linear level, it is possible to explicitly construct the glued geometry describing this situation through order ϵ^3. The metric is

$$
\begin{aligned}
ds^2 =& -rd\tau^2 + 2d\tau dr + dx_i dx^i \\
&- \big[2(1-r/r_c)v_i dx^i d\tau + (1-r/r_c)\left(\partial_j v_i + \partial_i v_j\right) dx^i dx^j \\
&\qquad -2\left(r - \frac{r^2}{2r_c} - r_c/2\right)\partial^2 v_i dx^i d\tau\Big] \\
&- \delta(\tau-\tau_*)\left[\left(4(1-r/r_c)F_i + \frac{2}{r_c}\alpha_i\right)dx^i d\tau - \frac{2}{r_c}\beta_{ij}dx^i dx^j\right] + \ldots
\end{aligned}
\tag{4.20}
$$

where F_i is an arbitrary function of x^i obeying $\partial_i F^i = 0$. β_{ij} and α_i (which is divergence free) are both functions of x^i and related to F_i by

$$
\partial^j \partial_j \alpha_i = F_i, \qquad \beta_{ij} = \partial_i \alpha_j + \partial_j \alpha_i. \tag{4.21}
$$

Since the metric on Σ_c is no longer flat, the constraint equations become linearized Navier-Stokes with a forcing term similar to that described in [37]. For this configuration we have

$$
\partial_\tau v^i - \eta \partial^2 v^i = F^i(x)\delta(\tau - \tau_*). \tag{4.22}
$$

Clearly, since $v_i(x,\tau)$ is taken to vanish for $\tau < \tau_*$, the forcing term will cause it to jump to $F_i(x^i)$ at $\tau = \tau_*$, after which it will evolve according to Navier-Stokes. Given (4.22) this geometry solves the linearized Einstein equations everywhere, and is characterized by an arbitrary divergence-free vector field $F_i(x)$. Before $\tau = \tau_*$ it is flat, while afterward it is, up to a coordinate transformation, the linearization of (4.12).

At the nonlinear level, the equations are cumbersome and we have been unable to explicitly construct the analog of (4.20) away from Σ_c. However it seems plausible that qualitatively similar solutions persist at the nonlinear level.

4.3.3 Singularities at $r = \pm\infty$

The square of the Riemann tensor for the solution (4.12) is given by

$$\mathcal{R}^2 = -\frac{3}{2r_c^2}\left(\partial_i v_j - \partial_j v_i\right)^2 - 2\frac{r}{r_c^2}\left[\partial^2 v_i \partial^2 v^i + 3\partial^i v^j \left(\partial_j \partial^2 v_i - \partial_i \partial^2 v_j\right)\right] + \ \ldots . \tag{4.23}$$

This expression diverges at $r = \pm\infty$. Of course perturbation theory cannot be trusted when $|r|$ is of order $\frac{1}{\epsilon}$, so the computation is unreliable in this regime. Whether or not there are actual divergences in these regions will depend on the details of the solution. In general, at $r = -\infty$, black hole type singularities may plausibly arise.

The divergence at $r = +\infty$ is outside the cutoff surface, so a priori need not concern us. Still we may ask what happens if we try to extend the solution to this region. In general relativity with no cosmological constant it is hard to find solutions which are asymptotically flat in codimension one: i.e. there are no codimension one black holes. This suggests that many configurations will be singular if extended to $r = +\infty$. On the other hand, if we add a negative cosmological constant, there are codimension one asymptotically AdS black holes. At large r the cosmological term tends to dominate, and we expect in this case many solutions to have nonsingular extensions to this region. However, as we will see below, the hydrodynamic regime is small r so the large r behavior is of limited interest for the present purposes.

4.3.4 Uniqueness

Equation (4.12) gives the first three orders in the ϵ-expansion of metrics satisfying the Einstein equations with the prescribed boundary data. These solutions are constructed from nonlinear solutions of the incompressible Navier-Stokes equations. The latter are in turn, assuming existence and uniqueness for Navier-Stokes, specified by a divergence-free vector field $v^i(x, \tau_*)$ at a moment of time τ_*.

One may ask whether or not (4.12) is the unique solution with the prescribed boundary data (up to coordinate transformations and field redefinitions) associated to a given $v^i(x, \tau_*)$. This can be addressed in the context of a combined weak-field expansion and ϵ-expansion. The problem was solved to leading nontrivial order in the weak-field expansion in [37]. The unique solution is the first two lines of (4.12), but with a v^i obeying the linearized Navier-Stokes equation. Generally one does not expect the dimension of the solution space in weak-field perturbation theory to change unless there is a linearization instability and associated obstruction. In the present case, the only potential obstruction is the Navier-Stokes equation which we are assuming can be solved. Hence one expects the solution (4.12) to be unique at each order in the ϵ-expansion, up to the usual ambiguity of adding solutions of the leading order equations at subleading orders.

4.4 Alternate presentation

In this section we give an alternate presentation of the metric (4.12) in which all the factors of ϵ appear explicitly, without being hidden in the functional dependence on the coordinates. This is accomplished by first transforming to hatted coordinates

$$x^i = \frac{r_c \hat{x}^i}{\epsilon}, \qquad \tau = \frac{r_c \hat{\tau}}{\epsilon^2}, \qquad r = r_c \hat{r} \tag{4.24}$$

so that $\partial_{\hat{\tau}} = \mathcal{O}(\epsilon^0)$ and we denote $\hat{\partial}_i = \frac{\partial}{\partial \hat{x}^i} = \mathcal{O}(\epsilon^0)$. In the new coordinates

$$\begin{aligned} ds^2_{p+2} =& -\frac{\hat{r} r_c^3}{\epsilon^4} d\hat{\tau}^2 + \frac{2r_c^2}{\epsilon^2} d\hat{\tau} d\hat{r} + \frac{r_c^2}{\epsilon^2} d\hat{x}_i d\hat{x}^i \\ & - 2r_c^2 \frac{1-\hat{r}}{\epsilon^2} \hat{v}_i d\hat{x}^i d\hat{\tau} - 2r_c \hat{v}_i d\hat{x}^i d\hat{r} \qquad (4.25) \\ & + (1-\hat{r}) \left[r_c^2 \frac{\hat{v}^2 + 2\hat{P}}{\epsilon^2} d\hat{\tau}^2 + r_c \hat{v}_i \hat{v}_j d\hat{x}^i d\hat{x}^j \right] + r_c(\hat{v}^2 + 2\hat{P}) d\hat{\tau} d\hat{r} \\ & - (\hat{r}^2 - 1) r_c \hat{\partial}^2 \hat{v}_i d\hat{x}^i d\hat{\tau} + \ \ldots, \qquad (4.26) \end{aligned}$$

where $\hat{P}(\hat{x}, \hat{\tau}) = \frac{1}{\epsilon^2} P(x(\hat{x}), \tau(\hat{\tau}))$, $\hat{v}_i(\hat{x}, \hat{\tau}) = \frac{1}{\epsilon} v_i(x(\hat{x}), \tau(\hat{\tau}))$, $\hat{v}^2 \equiv \hat{v}_i \delta^{ij} \hat{v}_j$ and i, j indices are raised and lowered with δ_{ij}. The usual Navier-Stokes equation for v, P with $\eta = r_c$ implies

$$\partial_{\hat{\tau}} \hat{v}_j - \hat{\partial}^2 \hat{v}_j + \hat{v}^k \hat{\partial}_k \hat{v}_j + \hat{\partial}_j \hat{P} = 0. \qquad (4.27)$$

This is the Navier-Stokes equation with $\eta = 1$ and no factors of ϵ or r_c.

Finally let us consider the rescaled metric $d\hat{s}^2_{p+2} = \frac{\epsilon^2}{r_c^2} ds^2_{p+2}$. The Einstein tensor is invariant under such constant metric rescalings. Rearranging terms and defining

$$\lambda \equiv \tfrac{\epsilon^2}{r_c} \qquad (4.28)$$

one finds

$$\begin{aligned} d\hat{s}^2_{p+2} =& -\frac{\hat{r}}{\lambda} d\hat{\tau}^2 \\ & + \left[2 d\hat{\tau} d\hat{r} + d\hat{x}_i d\hat{x}^i - 2(1-\hat{r}) \hat{v}_i d\hat{x}^i d\hat{\tau} + (1-\hat{r})(\hat{v}^2 + 2\hat{P}) d\hat{\tau}^2 \right] \qquad (4.29) \\ & + \lambda \left[(1-\hat{r}) \hat{v}_i \hat{v}_j d\hat{x}^i d\hat{x}^j - 2\hat{v}_i d\hat{x}^i d\hat{r} + (\hat{v}^2 + 2\hat{P}) d\hat{\tau} d\hat{r} + (1-\hat{r}^2) \hat{\partial}^2 \hat{v}_i d\hat{x}^i d\hat{\tau} \right] + \ldots \end{aligned}$$

The Brown-York stress tensor is

$$\begin{aligned} \hat{T}^{\hat{\tau}}_{\hat{\tau}} &= -\sqrt{\lambda} \hat{v}^2 + \mathcal{O}(\lambda^{3/2}), \qquad \hat{T}^{\hat{\tau}}_i = -\sqrt{\lambda} \hat{v}_i + \mathcal{O}(\lambda^{3/2}), \\ \hat{T}^i_j &= \frac{1}{\sqrt{\lambda}} \delta^i_j + \sqrt{\lambda} \left[\hat{v}^i \hat{v}_j + \hat{P} \delta^i_j - 2\hat{\partial}^i \hat{v}_j \right] + \mathcal{O}(\lambda^{3/2}). \qquad (4.30) \end{aligned}$$

The important point here is that the geometry depends only on the ratio $\lambda = \frac{\epsilon^2}{r_c}$ and not ϵ or r_c separately.

Given that the rescaled geometry depends only on λ and the ϵ-dependence (4.19) of the unrescaled geometry (4.12) we conclude that in the hatted coordinates

$$\begin{aligned} G_{\hat{\tau}\hat{\tau}} &= \frac{r_c^2}{\epsilon^4} G_{\tau\tau} \sim \mathcal{O}(\lambda^0), \quad G_{\hat{i}\hat{j}} = \frac{r_c^2}{\epsilon^2} G_{ij} \sim \mathcal{O}(\lambda), \\ G_{\hat{r}\hat{\tau}} &= \frac{r_c^2}{\epsilon^2} G_{rr} \sim \mathcal{O}(\lambda), \quad G_{\hat{r}\hat{r}} = r_c^2 G_{rr} \sim \mathcal{O}(\lambda^2) \\ G_{\hat{\tau}\hat{i}} &= \frac{r_c^2}{\epsilon^3} G_{\tau i} \sim \mathcal{O}(\lambda^{1/2}), \quad G_{\hat{r}\hat{i}} = \frac{r_c^2}{\epsilon} G_{ri} \sim \mathcal{O}(\lambda^{3/2}). \end{aligned} \tag{4.31}$$

Given the explicit factor of λ^{-1} in $g_{\hat{\tau}\hat{\tau}}$, it is not immediately obvious in this presentation that in a direct computation the Einstein tensor will even have a good Taylor expansion in λ. What happens is that, because $g_{\hat{r}\hat{r}} = 0$, there are only a limited number of powers of $g_{\hat{\tau}\hat{\tau}}$ that can appear in the Einstein tensor, and one may thereby directly recover (4.31). In fact, direct computation reveals we do slightly better; the last line may be replaced by

$$G_{\hat{\tau}\hat{i}} = \frac{r_c^2}{\epsilon^3} G_{\tau i} \sim \mathcal{O}(\lambda^1), \quad G_{\hat{r}\hat{i}} = \frac{r_c^2}{\epsilon} G_{ri} \sim \mathcal{O}(\lambda^2). \tag{4.32}$$

Notice that $G_{\hat{\tau}\hat{\tau}}$ in (4.31) is of order λ^0 rather than λ^1. We can improve this by computing a few higher order pieces of the metric. Specifically, we add to (4.29)

$$-2\lambda(1-\hat{r})\hat{q}_i d\hat{x}^i d\hat{\tau} + 2\lambda^2 g_{\hat{r}i}^{(2)} d\hat{r} d\hat{x}^i + \lambda^2 g_{ij}^{(2)} d\hat{x}^i d\hat{x}^j + \ \ldots \tag{4.33}$$

Demanding that the r-independent pieces of $G_{\hat{\tau}\hat{\tau}} = 0$ be solved through order λ^0 then fixes $\hat{q}_i(\hat{\tau}, \hat{x})$:

$$\hat{\partial}_i \hat{q}^i = \hat{\partial}^2 \hat{v}^2 - \frac{1}{2}\hat{v}^i \hat{\partial}_i \hat{v}^2 - \frac{3}{2}\partial_{\hat{\tau}} \hat{v}^2 - \frac{1}{2}\left(\hat{\partial}_i \hat{v}_j + \hat{\partial}_j \hat{v}_i\right)^2. \tag{4.34}$$

Apparently $\hat{q}_i$ is a kind of heat current. Demanding that the entire $G_{\hat{\tau}\hat{\tau}} = 0$ through order

λ^0 gives us a differential equation for the combination $\hat{Q}(\hat{r},\hat{\tau},\hat{x}) \equiv -2\hat{\partial}^i g^{(2)}_{\hat{r}i} + \partial_{\hat{r}} g^{i}{}_{i}^{(2)}$:

$$\hat{Q} + 2\hat{r}\partial_{\hat{r}}\hat{Q} = 2\hat{\partial}_i\hat{q}^i - 2\hat{v}_i\hat{q}^i + 3\hat{r}\hat{\partial}^2\hat{v}^2 - \frac{\hat{r}}{2}\left(\hat{\partial}_i\hat{v}_j + \hat{\partial}_j\hat{v}_i\right)^2 \\ + 2\hat{\partial}_i\hat{v}_j\hat{\partial}^j\hat{v}^i - \hat{v}^j\hat{\partial}_j\hat{v}^2 + \left(\hat{v}^2\right)^2 - 2\hat{v}^i\hat{\partial}_i\hat{P} + 2\hat{P}\hat{v}^2. \tag{4.35}$$

Choosing $\hat{q}_i$, $\hat{Q}$ accordingly, we find that as desired all components of the Einstein equations vanish for $\lambda \to 0$:

$$G_{\hat{r}\hat{a}}, G_{\hat{a}\hat{b}}, G_{\hat{r}\hat{r}} = \mathcal{O}(\lambda). \tag{4.36}$$

4.5 Petrov type

Interestingly, this geometry is of an algebraically special type. We consider the case of $p = 2$ to connect to the well-studied Petrov classification of spacetimes [59]). A geometry is Petrov type II if there exists a real null vector k^μ such that the Weyl tensor satisfies

$$W_{\mu\nu\rho[\sigma}k_{\lambda]}k^\nu k^\rho = 0. \tag{4.37}$$

This happens if the invariant $I^3 - 27J^2$ vanishes where I, J are both specific combinations of Weyl tensor components which can be found in [59]. For the metric (4.12), the lowest nonzero entries for I, J are at $\mathcal{O}(\epsilon^4)$ and $\mathcal{O}(\epsilon^6)$ respectively. Hence, the first contribution to the invariant would be at $\mathcal{O}(\epsilon^{12})$; however the invariant vanishes through $\mathcal{O}(\epsilon^{13})$. At higher order in ϵ, it gets modified by corrections to (4.12); we expect that including higher order terms in (4.12) enables (4.37) to be satisfied exactly.

4.6 Nonlinear solution in the near-horizon expansion

In section 4, the nonlinear Einstein equations with certain boundary conditions were solved in the non-relativistic, long-wavelength hydrodynamic ϵ-expansion. This generalized the analysis given in [37] of the ϵ-expansion for linearized modes. [37] also considered, for linearized modes, a second, near-horizon expansion. Although physically inequivalent, the two expansions were found to be equivalent mathematically and reduce to the linearized dynamics of an incompressible fluid. In this section, we consider the nonlinear version of the near-horizon expansion and find that it is again mathematically equivalent to the nonlinear ϵ-expansion.

In the ϵ-expansion one solves the shooting problem for long-wavelength perturbations of Σ_c with a fixed leading-order extrinsic curvature. The proper acceleration of a worldline at fixed x^i in Σ_c is to leading order just proportional to $K_{\tau\tau}$, so we may also view this as fixing the acceleration away from the origin. In the near-horizon expansion, instead of expanding in the wavelength one expands in the inverse acceleration. We begin with the flat metric on the Rindler wedge

$$ds^2_{p+2} = -r d\tau^2 + 2d\tau dr + dx_i dx^i. \tag{4.38}$$

To avoid confusion with the notation of the previous section we put the boundary on the accelerating surface

$$r = \tilde{r}_c, \tag{4.39}$$

so that $r \leq \tilde{r}_c$. The near-horizon, large acceleration, limit is $\tilde{r}_c \to 0$. In order to exhibit the $\tilde{r}_c$-dependence explicitly in the metric we transform to $r = \tilde{r}_c \hat{r}$, $\tau = \frac{\hat{\tau}}{\tilde{r}_c}$ so that $r \leq 1$ and

$$ds^2_{p+2} = -\frac{\hat{r}}{\tilde{r}_c} d\hat{\tau}^2 + 2d\hat{\tau} d\hat{r} + dx_i dx^i. \tag{4.40}$$

In these coordinates the near-horizon limit rescales to infinity the coefficient of $d\hat{\tau}^2$ at any finite $\hat{r}$.

We now wish to consider perturbations of this metric solving the Einstein equations order by order in the near-horizon expansion parameter $\tilde{r}_c$ that are consistent with a flat induced metric at $\hat{r} = 1$. At the level of linear perturbations, the most general solution was found in [37] (characterized in terms of the data at $r = \tilde{r}_c$). This solution is (for all r)

$$\begin{aligned} d\hat{s}^2_{p+2} =& -\frac{\hat{r}}{\tilde{r}_c} d\hat{\tau}^2 + 2d\hat{\tau}d\hat{r} + dx_i dx^i - 2(1-\hat{r})v_i dx^i d\hat{\tau} \\ &+ \tilde{r}_c\left[(1-\hat{r}^2)\partial^2 v_i dx^i d\hat{\tau} - 2v_i dx^i d\hat{r}\right] + \mathcal{O}(\tilde{r}_c^2) \end{aligned} \tag{4.41}$$

where $\partial_i v^i = 0$ and $\partial_{\hat{\tau}} v^i - \partial^2 v^i = 0$. That is, v^i is an incompressible fluid flow obeying the linearized Navier-Stokes equation with unit kinematic viscosity.

The nonlinear generalization of (4.41) which solves the nonlinear Einstein equations to $\mathcal{O}(\tilde{r}_c)$ is

$$\begin{aligned} d\hat{s}^2_{p+2} =& -\frac{\hat{r}}{\tilde{r}_c} d\hat{\tau}^2 \\ &+ \left[2d\hat{\tau}d\hat{r} + dx_i dx^i - 2(1-\hat{r})v_i dx^i d\hat{\tau} + (1-\hat{r})(v^2 + 2\hat{P})d\hat{\tau}^2\right] \\ &+ \tilde{r}_c\big[(1-\hat{r})v_i v_j dx^i dx^j - 2v_i dx^i d\hat{r} + (v^2 + 2\hat{P})d\hat{\tau}d\hat{r} \\ &+ (1-\hat{r}^2)\partial^2 v_i dx^i d\hat{\tau} - 2(1-\hat{r})\hat{q}_i(\hat{\tau},\hat{r},x)dx^i d\hat{\tau}\big] \\ &+ \tilde{r}_c^2\left[2g^{(2)}_{\hat{r}i}(\hat{\tau},\hat{r},x)dx^i d\hat{r} + g^{(2)}_{ij}(\hat{\tau},\hat{r},x)dx^i dx^j\right] + \mathcal{O}(\tilde{r}_c^2) \end{aligned} \tag{4.42}$$

provided $\partial_i v^i = 0$, $\partial_{\hat{\tau}} v_j - \partial^2 v_j + v^k \partial_k v_j + \partial_j \hat{P} = 0$. $\hat{q}_i$, $g^{(2)}_{\hat{r}i}$, $g^{(2)}_{ij}$ are solutions of first order differential equations of the type (4.34) and (4.35). Further $\mathcal{O}(\tilde{r}_c^2)$ pieces do not affect the equations of motion to this order.

We can now see explicitly that making the notation change $v \to \hat{v}$, $x^i \to \hat{x}^i$ and $\tilde{r}_c \to \lambda$

in (4.42) gives us the rescaled solution (4.29) in section 6. Hence the near-horizon and hydrodynamic expansions are mathematically equivalent.

Since we are identifying $\tilde{r}_c = \lambda = \frac{\epsilon^2}{r_c}$, $r_c \to \infty$ in the metric (4.12) is actually equivalent to the near-horizon limit $\tilde{r}_c \to 0$ in (4.42). This may at first seem odd, but the near-horizon-hydrodynamic equivalence involves a constant rescaling of (4.12) by a factor of $\frac{1}{r_c^2}$, the proper distance to the cutoff surface in the rescaled metric (4.42) indeed behaves as $\frac{1}{\sqrt{r_c}}$.

Chapter 5

From Petrov- Einstein to Navier-Stokes

5.1 Introduction

As an intriguing aside, it was further noted in previous chapter that for the four-dimensional case the geometry so constructed is, at least at leading nontrivial order in λ, of an algebraically special variety known as restricted Petrov type [58, 59].[1] In this chapter we turn the logic around and show, in every dimension, that imposing a Petrov type I condition in suitable circumstances reduces the Einstein equation to the Navier-Stokes equation in one lower dimension. Hence regularity on the future horizon and the Petrov type I condition are equivalent (at least) as far as the universal scaling behavior is concerned.[2] However, as

[1] It would be interesting to understand if this algebraic specialty persists to all orders for some choice of higher-order boundary conditions.

[2] One way of understanding why there should be such an equivalence is that for $\lambda \to 0$ Σ_c approaches the horizon with null normal ℓ on the past portion $\mathcal{H}^-$. Our type I condition is then the vanishing of the Weyl tensor components $\ell^\gamma \ell^\delta C_{\alpha\gamma\beta\delta} = -\ell^\gamma \nabla_\gamma \sigma_{\alpha\beta} - \theta\sigma_{\alpha\beta}$, where $\sigma_{\alpha\beta}$ is the shear and θ the expansion of $\mathcal{H}^-$. This

we shall see, imposing the Petrov condition is mathematically much simpler than imposing regularity.

More specifically, we embed an intrinsically flat $p+1$-dimensional timelike hypersurface Σ_c into a $p+2$-dimensional solution of Einstein's equation. We then impose the Petrov type I condition, defined below, with respect to the ingoing and outgoing pair of null vectors whose tangents to Σ_c generate time translations.[3] This condition sets to zero a total $\frac{(p+2)(p-1)}{2}$ components of the Wely tensor. On Σ_c this constraint reduces the $\frac{(p+1)(p+2)}{2}$ components of the extrinsic curvature K_{ab} to $p+2$ unconstrained variables, which may be interpreted as the energy density, velocity field v^i and pressure P of a fluid living on the hypersurface. Simply put, this Petrov condition reduces gravity to a fluid. The $p+2$ Einstein constraint equations on Σ_c then become an equation of state and evolution equation for the fluid variables. These highly nonlinear fluid equations are not, to the best of our knowledge, anything previously encountered in fluid dynamics. However, we next consider an expansion around a limit where Σ_c is highly accelerated, i.e. the mean curvature K diverges. At leading order in this expansion, the constraint equations are shown to reduce exactly to the incompressible nonlinear Navier-Stokes equation for v^i and P and the leading-order extrinsic geometry of Σ_c evolves as an incompressible fluid. Hence the Petrov type I condition has the holographic character of relating a theory of gravity in $p+2$ dimensions to a theory without gravity in $p+1$ dimensions.

in turn implies the shear in $\mathcal{H}^-$ vanishes. Equivalently there are no gravity waves passing through $\mathcal{H}^-$ and no incoming flux from the past. We thank Thibault Damour for discussions on this point.

[3] A space is said to be Petrov type I if there is some choice of null vectors with respect to which the Weyl tensor obeys certain identities described below. Due to special features of four dimensions, every 4D Einstein space is Petrov type I with respect to some null vectors, but not necessarily the ones related to time translations on Σ_c. Those that are Petrov type I with respect to these null vectors in fact also obey the stronger Petrov type II condition: this is the result quoted in [61].

In the appendix B we describe an an alternate set of boundary conditions on Σ_c, of possible interest in various contexts discussed therein, in which the mean curvature K is fixed. These are shown to differ only at subleading order and also lead to the universal incompressible Navier-Stokes equation in the near-horizon scaling limit.

5.2 Σ_c hypersurface geometry

We wish to consider the "initial" data on a timelike, $p+1$ dimensional hypersurface in a $p+2$-dimensional Einstein space.[4] We take the intrinsic metric to be flat

$$ds_{p+1}^2 = \eta_{ab}dx^a dx^b = -(dx^0)^2 + \delta_{ij}dx^i dx^j, \quad a, b = 0, ...p, \quad i, j = 1, ...p. \tag{5.1}$$

The extrinsic curvature K_{ab} is subject to the $p+1$ "momentum constraints"

$$\partial^a(K_{ab} - \eta_{ab}K) = 0, \tag{5.2}$$

as well as the "Hamiltonian constraint"

$$K_{ab}K^{ab} - K^2 = 0. \tag{5.3}$$

Satisfying these $p+2$ constraints reduces the $\frac{(p+1)(p+2)}{2}$ components of K_{ab} to $\frac{(p-1)(p+2)}{2}$ locally undetermined variables.

Given the bulk Einstein equation

$$G_{\mu\nu} = 0, \quad \mu, \nu = 0, ...p+1, \tag{5.4}$$

[4] For a nice discussion of the geometrical structures relevant in the current context see [12, 13].

the Riemann and Weyl tensors are equal and determined on Σ_c. One finds the simple expressions for the projections to Σ_c

$$\begin{aligned} C_{abcd} &= K_{ad}K_{bc} - K_{ac}K_{bd} \\ C_{abc(n)} &= K_{bc,a} - K_{ac,b} \\ C_{a(n)b(n)} &= KK_{ab} - K_{ac}K^c_b \end{aligned} \tag{5.5}$$

Here $C_{abc(n)} \equiv C_{abc\mu}n^\mu$ etc. with n^μ the unit normal to Σ_c.

5.3 The type I constraint

In this section we describe the Petrov type I condition in $p+2$ dimensions [62, 63]. We first introduce the $p+2$ Newman-Penrose-like vector fields

$$\ell^2 = k^2 = 0, \;\; (k,\ell) = 1, \;\; (m_i, k) = (m_i, \ell) = 0, \;\; (m_i, m_j) = \delta_{ij}. \tag{5.6}$$

The spacetime is Petrov type I if for some choice of frame

$$C_{(\ell)i(\ell)j} \equiv \ell^\mu m_i^\nu \ell^\alpha m_j^\beta C_{\mu\nu\alpha\beta} = 0 \tag{5.7}$$

Now let us choose

$$m_i = \partial_i, \;\; \sqrt{2}\ell = \partial_0 - n, \;\; \sqrt{2}k = -\partial_0 - n \tag{5.8}$$

where n is the spacelike unit normal and ∂_i, ∂_0 the tangent vectors to Σ_c. Note that this choice singles out a preferred time coordinate and thus breaks Lorentz invariance of Σ_c. Using (5.5) the type I condition for this frame choice is

$$2C_{(\ell)i(\ell)j} = (K - K_{00})K_{ij} + 2K_{0i}K_{0j} + 2K_{ij,0} - K_{ik}K^k{}_j - K_{0i,j} - K_{0j,i} = 0 \tag{5.9}$$

Since the Weyl tensor is traceless, the type I condition imposes $\frac{p(p+1)}{2} - 1$ conditions on the $\frac{(p+1)(p+2)}{2}$ components of K_{ab}. We may think of it as determining the trace-free part of K_{ij}

in terms of K_{0i}, K_{00} and K. This leaves $p+2$ independent components, which is exactly the number of components of a compressible fluid with a local pressure, energy and momentum density. The Hamiltonian constraint

$$G_{\mu\nu}n^\mu n^\nu|_{\Sigma_c} = \frac{1}{2}(K^2 - K_{ab}K^{ab}) = 0 \tag{5.10}$$

can be viewed as an equation of state relating the pressure and energy density. The $p+1$ momentum constraint equations

$$G_{\mu b}n^\mu|_{\Sigma_c} = \partial^a K_{ab} - \partial_b K = 0, \tag{5.11}$$

where $G_{\mu b}$ denotes the projection of the second index onto Σ_c, are then the evolution equations for the fluid. Hence these $p+2$ constraints eliminate all local freedom on Σ_c, and reduce it to a boundary value problem on a p-dimensional initial spacelike slice of Σ_c.

Hence the Petrov condition has a holographic nature: it reduces a theory of gravity to a theory of a fluid without gravity in one less dimension. However, without any further expansion the fluid described here has rather exotic dynamical equations. In the next section we will see the dynamics became familiar when expanded around a large mean curvature limit.

5.4 The large mean curvature expansion

We now introduce a parameter λ into the general fluid solution and then expand in λ. The first step is to define

$$\tau = \lambda x^0 \tag{5.12}$$

so that

$$ds^2 = \eta_{ab}dx^a dx^b = -\tfrac{d\tau^2}{\lambda^2} + dx_i dx^i. \tag{5.13}$$

We describe the extrinsic geometry in terms of the stress tensor $t^a{}_b$ given in terms of $K^a{}_b$ by

$$t^\tau{}_\tau = K^j{}_j, \quad t = pK - \frac{p}{2\lambda}, \quad \hat{t}^i{}_j = -K^i{}_j - \text{trace}, \quad t^\tau{}_i = -K^\tau{}_i \tag{5.14}$$

where by construction $\hat{t}^i{}_i = 0$. We have separated out, in the definition of $t^a{}_b$, a constant "pressure" piece which will diverge as $\lambda \to 0$. When all other components in (5.14) except this diverging piece vanish, $K^\tau{}_\tau = \frac{1}{2\lambda}$ and Σ_c is then simply the hyperbola in the Rindler wedge of Minkowski space

$$ds^2 = -rdt^2 + 2dtdr + dx_i dx^i \ , \tag{5.15}$$

located at $r = \lambda^2$ (note $\tau = \lambda^2 t$). For $\lambda \to 0$ the mean curvature of Σ_c becomes large and it approaches its own future horizon. Hence the $\lambda \to 0$ limit can be thought of as a kind of near-horizon limit.

More generally, for finite λ, the type I conditions (5.9) written in terms of the variables (5.14) have the following form

$$\left(t^\tau{}_\tau - \frac{2}{p}(t - t^\tau{}_\tau) - \frac{1}{\lambda}\right)\hat{t}^i{}_j + \frac{2}{\lambda^2}t^{\tau i}t^\tau{}_j - \hat{t}^i{}_k\hat{t}^k{}_j - 2\lambda\hat{t}^i{}_{j,\tau} - \frac{2}{\lambda}\delta^{ki}t^\tau{}_{(k,j)} - \text{trace} = 0 \tag{5.16}$$

with i, j indexes raised and lowered with δ_{ij}. Now we expand in powers of λ taking $t^a{}_b \sim \mathcal{O}(\lambda^0)$ or smaller. That is, for the components appearing in (5.14)

$$t^a{}_b = \sum_{k=0}^{\infty} t^{a(k)}_{\ b}\lambda^k. \tag{5.17}$$

As there is only one term of order $\frac{1}{\lambda^2}$ in equation (5.16) it immediately implies that the leading term of $t^\tau{}_j \sim \mathcal{O}(\lambda)$ and the leading term of $\hat{\tau}^i{}_j$ is

$$\hat{t}^{(1)}_{ij} = 2t^{\tau(1)}_{\ i}t^{\tau(1)}_{\ j} - 2t^{\tau(1)}_{\ (i,j)} - \text{trace}. \tag{5.18}$$

The exact Hamiltonian constraint

$$(t^{\tau}{}_{\tau})^2 - 2\frac{(t^{\tau}{}_{i})^2}{\lambda^2} + t^{i}{}_{j}t^{j}{}_{i} - \frac{1}{\lambda}t^{\tau}{}_{\tau} - \frac{1}{p}t^2 = 0 \tag{5.19}$$

at leading order fixes $t^{\tau}{}_{\tau}$ as

$$t^{\tau(1)}_{\tau} = -2t^{\tau(1)}_{i}t^{\tau(1)}_{j}\delta^{ij}. \tag{5.20}$$

Finally we come to the momentum constraints

$$\partial^a t_{ab} = 0. \tag{5.21}$$

The time component gives at leading order

$$\partial^i t^{\tau(1)}_{i} = 0. \tag{5.22}$$

The space components are at leading order

$$\partial_\tau t^{\tau(1)}_{i} + 2t^{\tau(1)}_{k}\partial^k t^{\tau(1)}_{i} - \partial^2 t^{\tau(1)}_{i} + \frac{1}{p}\partial_i t^{(1)} = 0. \tag{5.23}$$

Identifying

$$t^{\tau(1)}_{i} = v_i/2, \qquad t^{(1)} = pP/2, \tag{5.24}$$

as the velocity and pressure fields, (5.22) and (5.23) become

$$\partial_k v^k = 0, \tag{5.25}$$

$$\partial_\tau v_i + v^k\partial_k v_i - \partial^2 v_i + \partial_i P = 0. \tag{5.26}$$

This is precisely the incompressible Navier-Stokes system in p space dimensions [64].

Chapter 6

Magnetohydrodynamics / Einstein-Maxwell correspondence

6.1 Introduction

When faced with the full complexity of the nonlinear Navier-Stokes equation, one may be tempted to start looking for a solution in the absence of external forces. However, there is a particular choice of forcing term arising from the Lorentz force which has been extensively studied in plasma physics. This suggests that it may be advantageous to establish a new version of the correspondence for the charged fluid, which would allow us to bring our knowledge of the magnetohydrodynamics (MHD) to bear on the problem of solving the Einstein-Maxwell equations. Previous attempts to establish such a correspondence [65] have considered background magnetic fields interacting with the fluid. Instead, we propose to examine dynamical magnetic fields (induced by the fluid's motion) without background electromagnetic fields.

As such, we work in the Rindler wedge of flat Minkowski space and investigate perturbations of the geometry in the hydrodynamic limit, subject to some boundary conditions. Our perturbation, which we carry out to third order, is parametrized by fluid fields which satisfy the MHD equations. Interestingly, we find that in this setup, the conductivity σ of the charged fluid is precisely the reciprocal of its fluid viscosity $\eta = 1/4\pi\sigma$. Similarly to [61], we show that the dual metric, after some suitable rescaling, depends on only one parameter, which is obtained from a combination of the derivative expansion parameter and the distance between the metric horizon and the fluid surface. It is therefore possible to translate the derivative expansion into the near-horizon expansion, and vice versa.

In some cases [61] the fluid/gravity duality involves metrics which are algebraically special. In four dimensions, we show that the metric dual to MHD obeys Petrov type II conditions up to third order.

The main results of the paper are in section 5, where we formulate the Cauchy problem for the Einstein-Maxwell equations, and describe one of its solutions in a hydrodynamic expansion. This analysis is preceded by a short review of the Einstein-Maxwell theory (section 2) and of the MHD equations (section 3) and their scaling properties (section 4). In section 6, we elaborate on our solution by constructing it order by order in perturbation theory. We supplement this presentation with sections 7 and 8, in which we provide some additional details about the near-horizon expansion and offer some checks for the Petrov type II. We conclude the paper with some thoughts, open questions and possible generalizations.

6.2 Einstein-Maxwell theory

The Einstein-Maxwell equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{6.1}$$

describe gravity coupled to the electromagnetic stress tensor

$$4\pi T_{\mu\nu} = F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F^2, \tag{6.2}$$

where the gauge field itself solves the Maxwell equations

$$\nabla_\mu F^{\mu\nu} = 0. \tag{6.3}$$

In the remainder of this paper, we work in units where $8\pi G = 1$ and $c = 1$. Rather than working directly with the gauge field A_μ, it is more convenient to use the field strength $F_{\mu\nu}$ and impose the Bianchi constraint

$$\nabla_{[\lambda} F_{\mu\nu]} = 0. \tag{6.4}$$

These equations are well studied and several exact solutions are known. A famous example is y the Reissner-Nordström solution, which describes a spherically-symmetric charged object in an asymptotically flat 4-dimensional space. Other known solutions include planar charged objects obeying Anti-deSitter asymptotics, as well as less familiar gravitational-wave-like solutions with both metric and gauge field fluctuations. We will also be interested in the latter type of solution.

6.3 Magnetohydrodynamics

In $3+1$ dimensions, the MHD equations with finite conductivity σ take the following form [1]:

$$\begin{gathered}
\nabla \cdot B = 0, \qquad \nabla \cdot v = 0, \\
\nabla \times E = -\partial_t B, \\
\nabla \times B = 4\pi J = 4\pi\sigma(E + [v \times B]), \\
\partial_t v + (v \cdot \nabla)v + \nabla P - \eta \nabla^2 v = [J \times B].
\end{gathered} \tag{6.5}$$

The equations can be partially solved for the electric field E and current J. The remaining equations then form a nonlinear system describing a fluid with velocity v, subject to pressure P and a magnetic field B:

$$\begin{gathered}
\nabla \cdot B = 0, \qquad \nabla \cdot v = 0, \\
\partial_t B = \nabla \times [v \times B] + \frac{1}{4\pi\sigma}\nabla^2 B, \\
\partial_t v + (v \cdot \nabla)v + \nabla P - \eta \nabla^2 v = \frac{1}{4\pi}(B \cdot \nabla)B - \frac{1}{8\pi}\nabla\left(B^2\right).
\end{gathered} \tag{6.6}$$

As in the case of the incompressible Navier-Stokes equation, there are sufficiently many equations to determine all the variables. It will prove useful to rewrite the system (6.6) in terms of the electromagnetic fields f_{ij}, $f_{\tau i}$, for $i = 1, \ldots, 3$ ($E_i = f_{i\tau}$, $B_i = \frac{1}{2}\epsilon_{ijk} f^{jk}$)

$$\begin{gathered}
\partial_{[k} f_{ij]} = 0, \qquad \partial_i v^i = 0, \\
\partial_\tau f_{ij} = \partial_i f_{\tau j} - \partial_j f_{\tau i}, \qquad f_{\tau i} = -\frac{1}{4\pi\sigma}\partial_j f_{ij} - v^k f_{ki}, \\
\partial_\tau v_i + v^j \partial_j v_i + \partial_i P - \eta \partial^2 v_i + \partial^j \pi_{ji} = 0, \qquad \pi_{jk} = \frac{1}{4\pi}\left(f_{jl} f_{kl} - \frac{1}{4} f^2 \delta_{jk}\right).
\end{gathered} \tag{6.7}$$

The MHD equations (6.7) can be generalized to higher dimensions by assuming $i = 1, \ldots, p$.

6.4 Scaling properties

The Navier-Stokes equation (equation (6.7) with no electromagnetic field) is famous for its scaling property: simultaneous rescaling of the coordinates and fields

$$\begin{aligned} v(x,\tau) &\longrightarrow \epsilon v\big(\epsilon x, \epsilon^2 \tau\big)\,, \\ P(x,\tau) &\longrightarrow \epsilon^2 P\big(\epsilon x, \epsilon^2 \tau\big)\,, \end{aligned} \tag{6.8}$$

leaves the equation invariant while preserving the viscosity η. This scaling property is responsible for the universality of the NS equation in capturing the low energy dynamics of fluids. It may be extended to the MHD equations (6.7) by requiring that the electromagnetic field obey the following scaling relation:

$$\begin{aligned} f_{ij}(x,\tau) &\longrightarrow \epsilon f_{ij}\big(\epsilon x, \epsilon^2 \tau\big) \\ f_{i\tau}(x,\tau) &\longrightarrow \epsilon^2 f_{i\tau}\big(\epsilon x, \epsilon^2 \tau\big)\,. \end{aligned} \tag{6.9}$$

The scaling properties of the MHD equations allow us to write an ansatz for the bulk gauge field:

$$\begin{aligned} F_{ij} &= \epsilon F^0_{ij} + \epsilon^2 F^1_{ij} + \ldots \\ F_{i\tau} &= \epsilon^2 F^0_{ir} + \epsilon^3 F^1_{i\tau} + \ldots \end{aligned} \tag{6.10}$$

In order to ensure that the Bianchi identities hold at each expansion order independently (so that fields of different orders do not mix), the rest of the components should be chosen to be:

$$\begin{aligned} F_{ir} &= \epsilon^0 F^0_{ir} + \epsilon F^1_{ir} + \ldots \\ F_{r\tau} &= \epsilon^1 F^0_{r\tau} + \epsilon^2 F^1_{r\tau} + \ldots \end{aligned} \tag{6.11}$$

6.5 MHD/gravity correspondence

We can now describe the MHD/gravity correspondence in the simplest possible setup. The starting point is the flat Minkowski metric in $(p+2)$-dimensional space,

$$ds^2 = -r d\tau^2 + 2 d\tau dr + dx_i^2, \qquad i = 1, \ldots, p, \tag{6.12}$$

with no background electromagnetic field. The hyper surface Σ_c at fixed radius $r = r_c$, whose induced metric is flat, is the background space in which the fluid theory evolves[1]. We will now study a perturbative deformation of the metric and electromagnetic field which obeys the Einstein-Maxwell equations

$$G_{\mu\nu} = 2G\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F^2\right), \tag{6.13}$$

$$\nabla_\mu F^{\mu\nu} = 0, \qquad \nabla_{[\lambda}F_{\mu\nu]} = 0, \qquad \mu = r, \tau, 1, \ldots, p, \tag{6.14}$$

and the following boundary conditions:

Regularity at the horizon: both the field strength F and the metric are regular at $r = 0$.

Dirichlet boundary conditions: the induced metric on Σ_c is a flat Minkowski metric, and there is no induced charge nor current on Σ_c, i.e.

$$F^{ir}(r_c) = F^{\tau r}(r_c) = 0. \tag{6.15}$$

[1]The Brown-York stress tensor is diagonal and can be trivially identified with the fluid stress tensor at rest.

One of the solutions to the Cauchy problem is[2]

$$
\begin{aligned}
ds^2_{p+2} =& -r d\tau^2 + 2 d\tau dr + dx_i dx^i - 2\left(1-\frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\
&+ \left(1-\frac{r}{r_c}\right)\left[(v^2+2P)d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\
&- \frac{1}{16\pi p}\left(1-\frac{r}{r_c}\right)^2 f^2 d\tau^2 + \frac{1}{2\pi r_c}\left(1-\frac{r}{r_c}\right)\left(f_{ik} f_{jl}\delta^{kl} - \frac{1}{4}\delta_{ij} f^2\right) dx^i dx^j \\
&- \frac{(r^2-r_c^2)}{r_c}\partial^2 v_i dx^i d\tau + \mathcal{O}\left(\epsilon^3\right), \\
r_c F =& \frac{1}{2} f_{ij} dx^i \wedge dx^j + f_{i\tau} dx^i \wedge d\tau - \partial_j f_{ij} dx^i \wedge dr + \mathcal{O}\left(\epsilon^3\right).
\end{aligned}
\tag{6.16}
$$

This geometry, which is parametrized by the fluid fields f_{ij}, v_i, P and $f_{i\tau}$ (which depend only on x^i and τ), will satisfy the Einstein-Maxwell equations to order $\mathcal{O}(\epsilon^4)$ provided that the fluid fields satisfy the MHD equations,

$$
\begin{aligned}
\partial_\tau v_i + v^j \partial_j v_i + \partial_i\left(P - \frac{p+2}{16\pi p} f^2\right) - r_c \partial^2 v_i + \partial^j \pi_{ji} = 0, \\
\pi_{jk} = \frac{1}{4\pi}\left(f_{jl} f_{kl} - \frac{1}{4} f^2 \delta_{jk}\right), \\
\partial_i v^i = 0, \\
\partial_\tau f_{ij} = \partial_i f_{\tau j} - \partial_j f_{\tau i}, \\
\partial_{[k} f_{ij]} = 0, \\
f_{\tau i} = -r_c \partial_j f_{ij} - v^k f_{ki}.
\end{aligned}
\tag{6.17}
$$

Interestingly, π_{ij} is the lowest component of the electromagnetic energy momentum tensor on Σ_c in the ϵ-expansion. Moreover, the two diffusion constants which enter the MHD equations turn out to be equal:

$$
\eta = \frac{1}{4\pi\sigma} = r_c. \tag{6.18}
$$

[2]Details of the derivation are provided in the next section.

Perhaps this relation is not unexpected for such a simple background metric, since there are no dimensionless parameters for this ratio to depend on.

6.6 Solution

The Cauchy problem described in section 5 is generally hard to solve exactly. Nevertheless, due to the scaling properties of the MHD system, it is possible to construct a perturbative solution, as was done in [61]. The expansion assumes small perturbation size and slowly-varying spacetime dependence:

$$v_i \sim \mathcal{O}(\epsilon), \quad \partial_i \sim \mathcal{O}(\epsilon), \quad \partial_\tau \sim \mathcal{O}(\epsilon^2), \quad P \sim \mathcal{O}(\epsilon^2), \quad f_{ij} \sim \mathcal{O}(\epsilon), \quad f_{i\tau} \sim \mathcal{O}(\epsilon^2). \tag{6.19}$$

The problem may be simplified even further, as follows. At each given order in the expansion, we may divide the equations into two groups: constraint equations and propagating equations. The former depend only on the data from lower orders because of the extra spatial ∂_i and time derivatives ∂_τ present, whereas the latter fix the radial dependence of the new metric components introduced at the same order. The Navier-Stokes equation with magnetic forcing is a constraint equation which appears at third order $\mathcal{O}(\epsilon^3)$ so it can be written in terms of the metric solution at $\mathcal{O}(\epsilon^2)$ order.

In the remainder of this section, we will construct the geometry up to and including the $\mathcal{O}(\epsilon^2)$ order and describe the constraint equations at $\mathcal{O}(\epsilon^3)$ order.

6.6.1 Zeroth order

The background metric is flat Minkowski space, which solves Einstein's equations with no source terms:

$$ds^2 = -r d\tau^2 + 2 d\tau dr + dx_i dx^i. \tag{6.20}$$

At zeroth order $\mathcal{O}(\epsilon^0)$, there is one nontrivial Maxwell equation:

$$\partial_r \left(r F^0_{ri} \right) = 0. \tag{6.21}$$

The only solution that is regular at $r = 0$ is the trivial solution $F^0_{ri} = 0$.

To summarize, at zeroth order $\mathcal{O}(\epsilon^0)$, the solution is:

$$\begin{aligned} ds^2 &= -r d\tau^2 + 2 d\tau dr + dx_i dx^i + \mathcal{O}(\epsilon), \\ F &= \mathcal{O}(\epsilon). \end{aligned} \tag{6.22}$$

6.6.2 First order

Next, we wish to introduce a deformation of the metric parameterized by the fluid fields v^i and P. The simplest way to do this is to use small Lorentz boosts, as was done in [37], resulting in

$$ds^2 = -r d\tau^2 + 2 d\tau dr + dx_i dx^i - 2 \left(1 - \frac{r}{r_c} \right) v_i dx^i d\tau - 2 \frac{v_i}{r_c} dx^i dr + \mathcal{O}(\epsilon^2). \tag{6.23}$$

There are no corrections to the electromagnetic field since the background field vanished in the first place. At first order in ϵ, the Maxwell equations are:

$$\begin{aligned} \partial_r F^0_{\tau r} = 0 \qquad &\Longrightarrow \qquad r_c F^0_{\tau r} = Q^0(x, \tau), \\ \partial_r \left(r F^1_{ri} \right) = 0 \qquad &\Longrightarrow \qquad F^1_{ri} = 0, \\ \partial_r F^0_{ij} = 0 \qquad &\Longrightarrow \qquad r_c F^0_{ij} = f_{ij}(x, \tau). \end{aligned} \tag{6.24}$$

In the above, Q^0 can be interpreted as the charge density of the dual fluid. The only solution satisfies boundary condition (6.15) corresponds to $Q^0 = 0$.

In summary, the solution to first order $\mathcal{O}(\epsilon^1)$ is:

$$ds^2 = -rd\tau^2 + 2d\tau dr + dx_i dx^i - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr + \mathcal{O}\left(\epsilon^2\right), \tag{6.25}$$
$$r_c F = \frac{1}{2} f_{ij} dx^i \wedge dx^j + \mathcal{O}\left(\epsilon^2\right).$$

6.6.3 Second order

Note that the zeroth and first order solutions did not impose any constraints on the fluid fields v^i and f_{ij}. On the other hand, in order to solve the second order equations, we will need to impose some constraints on the fluid fields. In addition, we will have to introduce some extra fields such as $P(x,\tau)$ and $f_{\tau i}(x,\tau)$.

At second order $\mathcal{O}(\epsilon^2)$, the Maxwell equations are:

$$\begin{aligned}
\partial_r F^0_{i\tau} = 0 \qquad &\Longrightarrow \qquad r_c F^0_{i\tau} = f_{i\tau}(x,\tau), \\
\partial_{[k} F^0_{ij]} = 0 \qquad &\Longrightarrow \qquad \partial_{[k} f_{ij]} = 0, \\
\partial_r F^1_{ij} = 0 \qquad &\Longrightarrow \qquad r_c F^1_{ij} = f^1_{ij}(x,\tau), \\
\partial_r F^1_{\tau r} = 0 \qquad &\Longrightarrow \qquad F^1_{\tau r} = Q^1(x,\tau), \\
\partial_r\left(rF^2_{ri} + F^0_{\tau i}\right) + \partial_j F^0_{ji} = 0 \qquad &\Longrightarrow \qquad r_c F^2_{ri} = \partial_j f_{ij}.
\end{aligned} \tag{6.26}$$

Here, Q^1 is the next order correction to the charge density. In order to satisfy the boundary conditions at Σ_c, we must set $Q^1 = 0$ and also require that

$$F^{ri}(r_c) = 0 \qquad \Longrightarrow \qquad r_c F^2_{ri} + F^0_{\tau i} + v^k F_{ki} = 0, \tag{6.27}$$

which has the following solution:

$$f_{\tau i} = -r_c \partial_j f_{ij} - v^k f_{ki}. \tag{6.28}$$

The equation above is one of the MHD equations (6.7) that we obtained by solving the Einstein-Maxwell equations at second order $\mathcal{O}(\epsilon^2)$. Having obtained the field strength to this order, we can evaluate the stress tensor to second order as well:

$$\begin{gathered} r_c^2 F^2 = f^2 + \mathcal{O}\left(\epsilon^3\right), \qquad 4\pi r_c^2 T_{ij} = f_{li} f_{lj} - \frac{1}{4} f^2 \delta_{ij}, \\ 4\pi r_c^2 T_{r\tau} = -\frac{1}{4} f^2, \qquad 4\pi r_c^2 T_{\tau\tau} = \frac{r}{4} f^2, \\ 4\pi T_{rr} = \mathcal{O}\left(\epsilon^3\right), \qquad 4\pi T_{\tau i} = \mathcal{O}\left(\epsilon^3\right), \qquad 4\pi T_{ri} = \mathcal{O}\left(\epsilon^3\right). \end{gathered} \tag{6.29}$$

The nontrivial contributions to the stress tensor will backreact on the metric and produce additional terms of order $\mathcal{O}(\epsilon^2)$ in $g_{ij}^{(2)}$ and $g_{\tau\tau}^{(2)}$. For example, the (i, j) component of the Einstein equations will take the form

$$R_{ij} = -\frac{1}{2} \partial_r \left(r \partial_r g_{ij}^{(2)} \right) = 8\pi G \left(T_{ij} - \frac{1}{p} T^\mu_\mu \delta_{ij} \right). \tag{6.30}$$

Using the boundary condition on Σ_c, we can write the solution in the form

$$\begin{aligned} g_{ij}^{(2)} &= \frac{1}{2\pi r_c} \left(1 - \frac{r}{r_c}\right) \left(f_{li} f_{lj} - \frac{1}{2p} f^2 \delta_{ij} \right), \\ g_{\tau\tau}^{(2)} &= -\frac{2+p}{16\pi p} \left(1 - \frac{r}{r_c}\right)^2 f^2. \end{aligned} \tag{6.31}$$

As aforementioned, at this order in the expansion, we must introduce a constraint equation. In this case, it amounts to the requirement that the velocity field be divergence free:

$$\partial_i v^i = 0. \tag{6.32}$$

To summarize, at second order $\mathcal{O}(\epsilon^2)$, the solution is defined by:

$$
\begin{aligned}
ds^2_{p+2} =& -r d\tau^2 + 2 d\tau dr + dx_i dx^i - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\
&+ \left(1 - \frac{r}{r_c}\right)\left[\left(v^2 + 2P\right) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\
&+ \frac{1}{2\pi r_c}\left(1 - \frac{r}{r_c}\right)\left(f_{li} f_{lj} - \frac{1}{2p} f^2 \delta_{ij}\right) dx^i dx^j \\
&- \frac{2+p}{16\pi p}\left(1 - \frac{r}{r_c}\right)^2 f^2 d\tau^2 + \mathcal{O}(\epsilon^3), \qquad (6.33)\\
r_c F =& \frac{1}{2} f_{ij} dx^i \wedge dx^j + \frac{1}{2} f^1_{ij} dx^i \wedge dx^j + f_{i\tau} dx^i \wedge d\tau - \partial_j f_{ij} dx^i \wedge dr + \mathcal{O}(\epsilon^3), \\
f_{\tau i} =& -r_c \partial_j f_{ij} - v^k f_{ki}, \qquad \partial_{[k} f_{ij]} = 0, \\
\partial_i v^i =& 0.
\end{aligned}
$$

6.6.4 Third order

As in the second order case, at third order in the ϵ-expansion we must once again introduce new fields and impose additional constraints on the ones that were introduced at lower orders.

To be more precise, the equations which are tangent to Σ_c are constraint equations, while the remaining equations fix the radial dependence of the geometry at order $\mathcal{O}(\epsilon^3)$ in terms of the fluid fields. We illustrate this point in the context of the Bianchi identity:

$$
\begin{aligned}
\partial_r F^1_{i\tau} = -\partial_i F^1_{\tau r} \qquad &\Longrightarrow \qquad r_c F^1_{i\tau} = f^1_{i\tau}, \\
\partial_r F^2_{ij} = \partial_i F^2_{rj} - \partial_j F^2_{ri} \qquad &\Longrightarrow \qquad r_c F^2_{ij} = r(\partial_i \partial_k f_{jk} - \partial_j \partial_k f_{jk}) + f^2_{ij}, \qquad (6.34)\\
\partial_{[k} F^1_{ij]} = 0 \qquad &\Longrightarrow \qquad \partial_{[k} f^1_{ij]} = 0, \\
\partial_\tau F^0_{ij} = \partial_i F^0_{\tau j} - \partial_j F^0_{\tau i} \qquad &\Longrightarrow \qquad \partial_\tau f_{ij} = \partial_i f_{\tau j} - \partial_j f_{\tau i}.
\end{aligned}
$$

The equations in the last two lines are tangent to Σ_c and therefore impose constraints on the fluid data $f_{ij}, f^1_{ij}, f_{i\tau}$ from lower orders. On the other hand, the equations in the first two lines fix the $\mathcal{O}(\epsilon^3)$ radial dependence of the field strength components in terms of the newly introduced fluid fields.

The Einstein-Maxwell constraint equations on Σ_c have the following form:

$$\begin{aligned} n_\mu \nabla_\nu F^{\mu\nu} &= 0, \\ n^\nu G_{\mu\nu} &= 8\pi G\, n^\nu T_{\mu\nu}, \\ n^\nu n^\mu G_{\mu\nu} &= 8\pi G\, n^\nu n^\mu T_{\mu\nu}, \end{aligned} \tag{6.35}$$

where n^μ is a unit normal vector to Σ_c. At third order $\mathcal{O}(\epsilon^3)$, the Maxwell constraint is

$$\partial_i \left(r F^2_{ri} + v^k F^0_{ki} + F^0_{\tau i} \right) + \partial_\tau F^0_{\tau r} = 0. \tag{6.36}$$

This condition is trivially satisfied due to our choice of boundary condition (6.15). The only nontrivial gravitational constraint at third order $\mathcal{O}\left(\epsilon^3\right)$ is:

$$\begin{aligned} 0 &= n^\mu G_{\mu i} - 8\pi G\, n^\mu T_{\mu i} \\ &= \frac{1}{2r_c}\left[\partial_\tau v_i + v^j \partial_j v_i + \partial_i P - r_c \partial^2 v_i + \frac{1}{4\pi}\partial^j \left(f_{jl} f_{il} - \frac{p+1}{2p} f^2 \delta_{ij} \right) \right]. \end{aligned} \tag{6.37}$$

It can be identified with the last of the MHD equations (6.7) after performing the redefinition

$$P - \left(\frac{p+2}{16\pi p} \right) f^2 \quad \longrightarrow \quad P. \tag{6.38}$$

Having established the solvability of the constraint equations at $\mathcal{O}(\epsilon^3)$ order, the Cauchy theorem applied to the Einstein-Maxwell equations guarantees the existence of a solution for the entirety of the $\mathcal{O}(\epsilon^3)$ equations. This full solution differs from (6.16) in that it contains additional fluid field terms at order $\mathcal{O}(\epsilon^3)$. Finally, note that we may choose $f^1_{ij} = 0$, which trivially satisfies the Bianchi identity. This concludes the derivation of the duality proposed in section 5.

6.7 Near-horizon expansion

In this section, we will establish the equivalence of the hydrodynamic expansion for the metric (6.16) to the near-horizon expansion of the geometry. In order to achieve this, we begin by performing the coordinate redefinition from [61], namely:

$$x^i = \frac{r_c}{\epsilon}\hat{x}^i, \qquad \tau = \frac{r_c}{\epsilon^2}\hat{\tau}, \qquad r = r_c\hat{r}, \tag{6.39}$$

In these new coordinates, the derivatives are no longer assumed to be small, i.e. $\hat{\partial}_i = \mathcal{O}(\epsilon^0), \partial_{\hat{\tau}} = \mathcal{O}(\epsilon^0)$, and the dual metric (6.16) takes the following form

$$\begin{aligned}
\frac{\epsilon^2}{r_c^2}ds^2_{p+2} = &-\frac{\hat{r}}{\lambda}d\hat{\tau}^2 + 2d\hat{\tau}d\hat{r} + d\hat{x}_i d\hat{x}^i - 2\left(1-\hat{r}\right)\hat{v}_i d\hat{x}^i d\hat{\tau} \\
&+ \left(1-\hat{r}\right)\left(\hat{v}^2 + 2\hat{P}\right)d\hat{\tau}^2 - \frac{1}{16\pi p}\left(1-\hat{r}\right)^2 \hat{f}^2 d\hat{\tau}^2 \\
&+ \lambda\Bigg[\left(1-\hat{r}\right)\left(\hat{v}_i\hat{v}_j + \frac{1}{2\pi}\hat{f}_{ik}\hat{f}_{jl}\delta^{kl} - \frac{1}{4\pi p}\hat{f}^2\delta_{ij}\right)d\hat{x}^i d\hat{x}^j - 2\hat{v}_i d\hat{x}^i d\hat{r} \\
&\qquad + \left(\hat{v}^2 + 2\hat{P}\right)d\hat{\tau}d\hat{r} + \left(1-\hat{r}^2\right)\hat{\partial}^2\hat{v}_i d\hat{x}^i d\hat{\tau}\Bigg] + \dots \\
\frac{\epsilon}{r_c}F = &\frac{1}{2}\hat{f}_{ij}d\hat{x}^i \wedge d\hat{x}^j + \hat{f}_{i\tau}d\hat{x}^i \wedge d\hat{\tau} - \lambda\hat{\partial}_j\hat{f}_{ij}d\hat{x}^i \wedge d\hat{r} + \dots
\end{aligned} \tag{6.40}$$

where we introduced a new expansion parameter $\lambda = \frac{\epsilon^2}{r_c}$ as well as new fluid fields defined by

$$\begin{aligned}
\hat{v}_i(\hat{x},\hat{\tau}) &= \frac{1}{\epsilon}v_i(\hat{x}(x),\hat{\tau}(\tau)), \qquad \hat{P}(\hat{x},\hat{\tau}) = \frac{1}{\epsilon^2}P(\hat{x}(x),\hat{\tau}(\tau)), \\
\hat{f}_{ij}(\hat{x},\hat{\tau}) &= \frac{1}{\epsilon}f_{ij}(\hat{x}(x),\hat{\tau}(\tau)), \qquad \hat{f}_{i\tau}(\hat{x},\hat{\tau}) = \frac{1}{\epsilon^2}f_{i\tau}(\hat{x}(x),\hat{\tau}(\tau)).
\end{aligned} \tag{6.41}$$

After a suitable rescaling, the geometry (6.40) will no longer depend on the two independent parameters r_c and ϵ; rather, it will be parameterized by the single parameter λ. Likewise, the r_c dependence also drops out of the MHD equations, which become:

$$\begin{aligned}
\partial_{\hat{\tau}}\hat{v}_i + \hat{v}^j\hat{\partial}_j\hat{v}_i + \hat{\partial}_i\left(\hat{P} - \frac{p+2}{16\pi p}\hat{f}^2\right) - \hat{\partial}^2\hat{v}_i + \frac{1}{4\pi}\hat{\partial}^j\left(\hat{f}_{jl}\hat{f}_{il} - \frac{1}{4}\hat{f}^2\delta_{ij}\right) = 0, \\
\hat{f}_{\tau i} = -\hat{\partial}_j\hat{f}_{ij} - \hat{v}^k\hat{f}_{ki}.
\end{aligned} \tag{6.42}$$

The distance between the metric horizon at $r = 0$ and the cutoff surface at $r = r_c$ in the rescaled metric (6.40) behaves as $\frac{1}{\sqrt{r_c}}$, so should not be surprising that there are two ways to make λ small: one way is to perform a hydrodynamic expansion in $\epsilon \ll 1$ on the fluid surface Σ_c while keeping r_c fixed; the other way consists of pushing the cutoff surface Σ_c close to the horizon ($r_c \gg 1$) while removing the small derivative restriction on the fluid fields (so that ϵ can be arbitrarily large).

6.8 Petrov type

As in [61], we find that in four dimensions ($p = 2$), the geometry (6.16) is of algebraically special Petrov type II, meaning that there exists a null vector k^μ such that the Weyl tensor satisfies

$$W_{\mu\nu\rho[\sigma}k_{\lambda]}k^\nu k^\rho = 0. \tag{6.43}$$

One may verify the existence of such a null vector by evaluating the invariant $I^3 - 27J^2$, which is a function of the metric. The details about I and J and their explicit value in terms of the metric components can be found in [59]. The lowest nontrivial components of I and J are typically of order $\mathcal{O}(\epsilon^4)$ and $\mathcal{O}(\epsilon^6)$, respectively. Hence we generally expect the invariant $I^3 - 27J^2$ to be of order $\mathcal{O}(\epsilon^{12})$, while an explicit computation for the invariant of the metric (6.16) reveals it to be of order $\mathcal{O}(\epsilon^{14})$.

6.9 Conclusion and open questions

The primary purpose of this section was to show that the fluid/gravity correspondence can be naturally extended to include electromagnetic fields, and to shed some light on this new facet of the duality.

We illustrated this new aspect of the correspondence in the simplest nontrivial background, namely the Rindler wedge of flat Minkowski space. In that context, we were able to obtain an explicit solution to the Einstein-Maxwell equations as a hydrodynamic expansion parameterized by the fluid fields with polynomial bulk dependence. In the process, we discovered that the dual MHD equations have equal magnetic and fluid diffusion constants.

In light of the results in [66], which were cast in a similar framework to ours [61], we believe that the Cauchy problem from section 5 admits a solution at all orders in the hydrodynamic expansion. In the 4-dimensional case, we were able to perform a test of the algebraically special character of the geometry, which turned out to be of Petrov type II. It is very likely that this statement will continue to hold in higher dimensions, though in such cases there is no analogue to the invariant I^3-27J^2 which can be used to perform the check. Nevertheless, it should be possible to generalize our solution to other background geometries. It seems worth investigating the dimensionless ratio of the two diffusion constants, as it might be subject to certain restrictions in the case of MHD theories with gravity duals. In particular, it would be interesting to find a background corresponding to the infinitely conducting fluid $\sigma=\infty$, which serves as a good approximation to real world MHD problems.

In [61], the observation that the metric was of an algebraically special type strongly suggested the hypothesis that algebraically special metrics have fluid duals [67]. The fact that the metric (6.16) is algebraically special leads us to formulate a new conjecture: Petrov type I metrics which solve the Einsten-Maxwell equations with properly aligned electromagnetic field strength appear to be dual to MHD-like fluid equations on codimension-one hypersurfaces. In the limit when the mean curvature of the hypersurface is large, these fluid equations reduce to the usual MHD equations; some work in this direction was done in [65].

Chapter 7

Possible applications and open questions

In previous chapters we have discussed some aspects of fluid/gravity correspondence in detailed and explicit form. As it usually happens the most interesting questions are hard to answer in such manner. In this chapter we want to discuss the most frequently asked questions related to the fluid gravity correspondence. The most popular questions are about the turbulence and possibility of transferring the known exact solutions in either fluid or gravity direction. We do not know a complete answer to either of the questions but we have something to say about both of them. We want to warn reader that this chapter contains some ongoing work and some results may change as we progress.

7.1 Exact solutions

There are known exact solutions for both the NS equations and the Einstein equation. The perturbative mapping that we described in chapter 4 in not very useful form mapping the exact solutions. If we start with the NS solution we will generate the perturbative series on the gravity side which typically neither terminate at some order or sum into something reasonable. If we start with known gravitational solution it typically takes a lot of effort to find a coordinate transformation to the ingoing EF coordinates that we using. The good illustration to this statement is a nice paper Bredberg and Strominger [68], where authors discussed a shear perturbations for the Schwarzchild black hole in four dimensions.

Many of the exact solutions to Einstein equations are algebraically special, what is not surprising because imposing some additional conditions on the Weyl tensor simplifies possible solutions. So given a type I solution we can try to find a hypersurface with large mean curvature and identify the Brown - York stress tensor with fluid stress tensor. However there are two major problems with such approach. It is hard to find metrics that are the type I and not higher type and typically the hypersurface would have rather general induced metric. The general induced metric means that the we will end up constructing a solution to the curved space modification of the NS equation with possible forcing terms. The higher algebraic type means that there are more constraints for the Weyl tensor which in turn implies that the velocities in NS equation are being further constrained. Let us illustrate the implications of both problems on a simple example: Kerr geometry.

7.2 Algebraically special types

In chapter 5 we showed that the type I condition imposed on the metric reduces the number of independent components of the Brown-York stress tensor from $\frac{(p+1)(p+2)}{2}$to $p+2$. The more special metrics (type II, III,...) have even more trivial Weyl tensor components. For example the type D geometries which represent a general rotating black holes in higher dimensions have the following nontrivial Weyl tensor components

$$\Psi^2_{ij} \equiv C_{ijk\ell}, \quad \Psi^2_{scalar} \equiv C_{k\ell k\ell}, \tag{7.1}$$

where we used the same null frame definitions as in chapter 5. For example a 4d Kerr black hole has as single complex component of the Weyl tensor which in Boyer Lindquist coordinates looks very simple:

$$\Psi^2_0 = -\tfrac{M}{6(r+ia\cos\theta)^3}, \tag{7.2}$$

with M being a back hole mass and a is the rotation parameter. Using our results from chapter 5 we can express the rest of Weyl tensor components. In particular we can use the type I condition and one of the type D conditions to obtain

$$C_{\ell i\ell j} = 0, \;\; \Psi^4_{ij} \equiv C_{kikj} = 0 \Rightarrow \partial_i v_j + \partial_j v_i + \mathcal{O}(\lambda^2) = 0. \tag{7.3}$$

Therefore the dual fluid should have trivial shear tensor σ_{ij}, what narrows possible fluid solutions to almost trivial ones. Similar story holds for the metrics of type II, but the explicit proof requires more work.

The vanishing shear $\sigma_{ij} = \partial_i v_j + \partial_j v_i$ is the Killing vector field condition for flat Euclidean space. If we allow for a nontrivial base space for a dual fluid the type D gravity solutions lead to the "Killing flows". In particular is the hypersurface Σ_c has induced metric of the form

$$ds^2 = -\tfrac{d\tau^2}{\lambda^2} + \gamma_{ij}(x)dx^i dx^j, \tag{7.4}$$

then the similar to chapter 5 analysis done by [65, 68] lead to the following generalization of the NS system

$$D_i v^i = 0, \quad \partial_\tau v_i + v^k D_k v_i - D^k(D_k v_i + D_i v_k) + \partial_i P' = D^k R^{(p)}_{ik}, \tag{7.5}$$

where D_i and $R^{(p)}_{ij}$ are covariant derivative and Ricci tensor for the the γ_{ij} metric. The forcing term can be removed by shifting pressure and using the Bianci identity

$$D^k(R^{(p)}_{ik} - \tfrac{1}{2}\gamma_{ik}R^{(p)}) = 0. \tag{7.6}$$

The NS equation (7.5) admits a static solution

$$v^i = k^i, \quad P = \tfrac{1}{2}k_i k_j \gamma^{ij}, \tag{7.7}$$

with k^i being a Killing symmetry of γ_{ij}. Thus we can conjecture that the type D solutions to the Einstein equations are dual to the Killing flow - solutions of the curved space NS equation. The evidence for such conjecture was provided by [68] where authors studied the perturbative deformations of the Schwarzchild black hole and corresponding dual fluid. The fluid was living on the S^2 and had the same couplings to the curvature as in (7.5). One of the solutions to such equation was Killing flow while the dual metric was the leading expansion of the slow rotating Kerr geometry. Another indirect evidence was provided by Minwalla and collaborators [38]. The showed that the the suitable expansion of the rotating black holes in higher dimensional AdS spaces coincides with the derivative expansion for the uniformly rotating relativistic fluid on the sphere of appropriate dimension.

7.3 Turbulence

Question about turbulence in our fluid dictionary is by far the most popular question after my talks on fluid/gravity, so the thesis would not be complete if I would not say some words about it. Generally the turbulent flow is hard to describe, but there are two regimes when we can use some symmetry arguments or/and perturbative approach. The first regime is developing the initial instability while the second is developed turbulence. Let us consider them separately.

7.3.1 Initial instability

Let us consider a uniform fluid flow with velocity u, parallel to the x-axis, incident from left to right on an infinite circular cylinder of diameter L with axis being along the z - direction[1]. For sufficiently small velocity the flow has the symmetries of the initial setup geometry (left-right, up-down, time translation, parallel translation along the cylinder's axis). If we increase velocity then at some point we may observe a left-right symmetry breaking and formation of small vortices behind the cylinder. The creating of such vortices is related to the instability of the particular solution to the nonlinear Navier-Stokes equation. The transition between the flows of different symmetries is controlled by the dimensionless Reynolds number (1.10)

$$Re = \tfrac{Lu}{\eta}. \tag{7.8}$$

Typically for the large $Re \sim 10^3$ we have many vortices and often a chaotic flow, while for $Re \ll 1$ we typically observe smooth flow with the symmetries of the initial setup.

We do not have much knowledge about nonlinear stability of the gravity solutions. The

[1] for more details and pictures you can can read [69, 70]

explicit theorems were proven only for some simple solutions with high amount of symmetry. We can try to use fluid/gravity correspondence to make some predictions for black hole nonlinear stability by mapping the black hole solution to the simple fluid flow and estimating its Reynold's number. However, the fluid dynamics only probing shear type of perturbations so the stability of dual fluid solution do not imply the same for gravity one.

Let us illustrate our proposition on the simple example of slowly rotating Kerr black hole in four dimensions. The dual fluid is a killing flow on the round two sphere at the horizon $L = r_h$, while the fluid velocity is proportional to the rotation parameter a. Therefore we can estimate the Reynolds number for this flow to be

$$Re_{Kerr} \sim \tfrac{a}{r_h} \sim \tfrac{J}{M^2}, \tag{7.9}$$

with J, M being angular momentum and the mass of the Kerr black hole. In our slow rotating approximation $Re_{Kerr} \ll 1$ so we do not expect any instabilities for the fluid flow and shear modes in gravity. This prediction agrees with the Kerr black hole stability analysis [71–73]. The Kerr solution develops a naked singularity when $J > M^2$, where $Re_{Kerr} > 1$, however we cannot trust our formula since we used slow rotation approximation to derive it. It might be interesting to study the dual fluid for near extremal Kerr black hole, solve for unstable fluid modes (if any) and compare the result to the $J > M^2$ bound. Another interesting proposition would be to consider Kerr-Newman solution where the absence of naked singularity requires

$$M^2 > a^2 + Q^2, \tag{7.10}$$

with Q being a balk hole charge. If the charge is close to the black hole mass $M - Q \ll M$ then even slow rotation may break the bound. We can still use the slow rotation approximation and expect a similar Killing flow on the two sphere at the horizon, while the other fluid parameters ρ, ν may have interesting dependence on dimensionless ratio Q/M.

7.3.2 Developed turbulence

Developed turbulence is chaotic fluid flow at very large Reynolds numbers and admits extra symmetries for statistical averages. Extra symmetries allow to fix triple velocity correlation function in the form of the Kolmogorov's 4/5 law [69]

$$\langle (\delta v(\ell))^3 \rangle = -\tfrac{4}{5}\bar{\epsilon}\ell, \tag{7.11}$$

where $\ell\delta v(\ell) \equiv \ell^i(v^i(x^i+\ell^i,\tau) - v^i(x^i,\tau))$ is longitudinal component of the velocity with $\bar{\epsilon}$ being mean energy dissipation per unit volume. The additional scaling symmetry of the NS equation that we discussed in chapter 3 allows to make a prediction for the two and higher point correlation functions of the velocity. Unfortunately we do not how to derive Kolmogorov's relation from the dual gravity solution, however there are some interesting implications of this relation to the black hole horizons. Adams, Chesler and Liu [74] showed that the perturbative gravity solution dual to the chaotic fluid with such correlation functions lead to the nontrivial fractal dimension of the black brane horizons in AdS.

Another interesting feature of the developed turbulence is so called energy cascading. The dissipation in the NS fluid is controlled by the viscous scale which is typically very small, while the energy input may happen at large scale. In the intermediate scale we can drop both viscous term and external force, so the energy is conserved and transferred from large to small scales. In our historical introduction chapter we mentioned that fluid/gravity correspondence was extensively discussed in the context of the AdS/CFT. The extra dimension for dual gravity solution often has an interpretation as a scale in dual theory, so it is reasonable to look at the radial dependence for our dual gravity solution. Unfortunately in both our constructions the NS equation is related to the constraint equation in gravity and therefore has frozen radial dependence. This statement follows from the fact that if constraint equations hold one hypersurface then they hold for any similar hypersurface.

7.3.3 More questions

The incompressible Navier - Stokes equation and the Einstein equations are probably the most studied equations, so any fluid/gravity relation can immediately be used to map interesting results, statements conjectures from one system to another. We hope that our work may be useful for better understanding of both Einstein and Navier Stokes equations.

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Appendix A

Singularies for ϵ^3 geometry

In the ϵ-expansion,

$$\begin{aligned}
ds^2_{p+2} = & - r d\tau^2 + 2 d\tau dr + dx_i dx^i - 2\left(1 - \frac{r}{r_c}\right) v_i dx^i d\tau - 2\frac{v_i}{r_c} dx^i dr \\
& + \left(1 - \frac{r}{r_c}\right)\left[(v^2 + 2P) d\tau^2 + \frac{v_i v_j}{r_c} dx^i dx^j\right] + \left(\frac{v^2}{r_c} + \frac{2P}{r_c}\right) d\tau dr \\
& + c_1 \log\left(\frac{r}{r_c}\right)\left(\partial_i v_j + \partial_j v_i\right) dx^i dx^j - \frac{(r^2 - r_c^2)}{r_c} \partial^2 v_i dx^i d\tau \\
& - 2\left(1 - \frac{r}{r_c}\right) q_i dx^i d\tau - 2c_1\left(r \log r - r_c \log r_c\right) \partial^2 v_i dx^i d\tau \\
& - 2c_1 \log\left(\frac{r}{r_c}\right) v^j \left(\partial_i v_j + \partial_j v_i\right) dx^i d\tau + 2c_1 \left(1 - \frac{r}{r_c}\right) v^j \partial_j v_i dx^i d\tau + \mathcal{O}(\epsilon^4)
\end{aligned} \tag{A.1}$$

solves the Einstein equations through $\mathcal{O}(\epsilon^3)$ if v_i obeys the incompressible Navier-Stokes equation with the "wrong" viscosity $\eta = r_c\,(1 + c_1)$ where c_1 is a nonzero constant. For this geometry, the square of the Riemann tensor is

$$\mathcal{R}^2 = -\frac{3}{2r_c^2}\left(\partial_i v_j - \partial_j v_i\right)^2 + \frac{c_1\,(c_1 + 2)}{r^2}\left[2\partial_i v_j \partial^j v^i + \frac{1}{2}\left(\partial_i v_j - \partial_j v_i\right)^2\right] + O(\epsilon^6) \tag{A.2}$$

which clearly diverges at $r = 0$ unless c_1 vanishes or $c_1 = -2$. The last possibility is the time reverse of the first and exponentially growing in the future.

Appendix B

Constant K boundary conditions

In this appendix we consider a modification of the flat "Dirichlet" boundary conditions $h_{ab} = \eta_{ab}$ imposed on the hypersurface Σ_c. In general there is freedom at higher orders in the choice of boundary conditions: any modification of the metric of order λ or smaller will not affect the universal emergence of the incompressible Navier-Stokes equation in the $\lambda \to 0$ scaling limit. The flat boundary conditions have been employed for their simplicity and naturalness. In this appendix we describe an alternate boundary condition for which the metric is only conformally flat and the mean curvature K is fixed to a constant. Roughly speaking this is Neumann rather than Dirichlet boundary conditions for the metric conformal factor.

These constant mean curvature boundary conditions are of interest for several reasons. Firstly, constant K hypersurfaces have interesting mathematical properties which have been the subject of much study over the last half century. In the present context they seem particularly appropriate because our expansion parameter is K^{-1}. Secondly, in recent generalizations to compact spherical horizons [68], a global obstruction (related to total energy conservation) appears at a subleading order which prevents one from fixing the total area of a spatial cross section of Σ_c. This obstruction is absent in the constant K formulation here which does allow the area to change.

We take the intrinsic metric of Σ_c to be conformally flat

$$ds^2_{p+1} = e^{2\rho}\eta_{ab}dx^a dx^b = e^{2\rho}(-(dx^0)^2 + dx_i dx^i), \tag{B.1}$$

where here and elsewhere i, j indices are raised and lowered with δ_{ij}. Instead of fixing $\rho = 0$ as above, we take constant mean curvature

$$K = e^{-2\rho}\eta^{ab}K_{ab} = \tfrac{1}{2\lambda} \tag{B.2}$$

It is convenient to describe the remaining components of the extrinsic geometry in terms of the conformally transformed, traceless stress tensor

$$\mathrm{T}_{ab} = e^{(p-1)\rho}K_{ab} - \tfrac{e^{(p+1)\rho}}{p+1}\eta_{ab}K, \tag{B.3}$$

in terms of which the the $p+1$ "momentum constraints" are

$$\partial^a \mathrm{T}_{ab} = 0, \tag{B.4}$$

The conformal factor ρ is then determined from the "Hamiltonian constraint" or York equation

$$-2p\partial_a\partial^a\rho + p(1-p)\partial_a\rho\partial^a\rho + e^{-2p\rho}\mathrm{T}_{ab}\mathrm{T}^{ab} - \frac{pe^{2\rho}}{4\lambda^2(p+1)} = 0, \tag{B.5}$$

with indices here raised and lowered with η. The Petrov type I condition for $\sqrt{2}\ell = e^{-\rho}\partial_0 - n$ is, instead of (5.9)

$$\begin{aligned} 2e^{2\rho}C_{\ell i\ell j} &= \tfrac{pe^{-(p-1)\rho}}{2\lambda(p+1)}\mathrm{T}_{ij} + e^{-2p\rho}(\mathrm{T}_{0i}\mathrm{T}_{0j} - \mathrm{T}_{00}\mathrm{T}_{ij} - \mathrm{T}_{aj}\mathrm{T}^a_i) \\ -\partial_i(e^{-p\rho}\mathrm{T}_{j0}) &- \partial_j(e^{-p\rho}\mathrm{T}_{i0}) + 2\partial_0(e^{-p\rho}\mathrm{T}_{ij}) + p\,\partial_i\partial_j\rho - p\,\partial_i\rho\partial_j\rho - \text{trace} = 0 \end{aligned} \tag{B.6}$$

To define the the large mean curvature expansion again take $\tau = \lambda x^0$ so that

$$ds^2 = e^{2\rho}\eta_{ab}dx^a dx^b = e^{2\rho}\left(-\tfrac{d\tau^2}{\lambda^2} + dx_i dx^i\right). \tag{B.7}$$

and instead of (5.14)

$$\mathrm{t}^\tau{}_\tau = \mathrm{T}^j{}_j + \frac{p}{2\lambda(p+1)}e^{(p+1)\rho}, \quad \hat{\mathrm{t}}^i{}_j = -\mathrm{T}^i{}_j - \text{trace}, \quad \mathrm{t}^\tau{}_i = -\mathrm{T}^\tau{}_i \tag{B.8}$$

where by construction $\hat{\mathrm{t}}^i{}_i = 0$. For these variables the type I conditions (B.6) written in terms of the variables (B.8) have the following form

$$\begin{gathered}-\tfrac{1}{\lambda}e^{\rho}\hat{\mathrm{t}}_{ij} + e^{-p\rho}[\tfrac{2}{\lambda^2}\mathrm{t}^\tau{}_i\mathrm{t}^\tau{}_j + \tfrac{p+2}{p}\mathrm{t}^\tau{}_\tau\hat{\mathrm{t}}_{ij} - \hat{\mathrm{t}}_{ik}\hat{\mathrm{t}}^k_j] - \tfrac{1}{\lambda}[(\partial_i - p\partial_i\rho)\mathrm{t}^\tau{}_j + (\partial_j - p\partial_j\rho)\mathrm{t}^\tau{}_i] \\ -2\lambda(\partial_\tau\hat{\mathrm{t}}_{ij} - p\partial_\tau\rho\;\hat{\mathrm{t}}_{ij}) + p\;\partial_i\partial_j\rho - p\;\partial_i\rho\partial_j\rho - \text{trace} = 0\end{gathered} \tag{B.9}$$

with i, j indexes raised and lowered with δ_{ij}. Now we expand in powers of λ taking $\mathrm{t}^a{}_b \sim \mathcal{O}(\lambda^0)$ We also so take $\rho \sim \mathcal{O}(\lambda)$ or smaller so that in the limit we recover a fluid in flat space. That is, for the components appearing in (B.8)

$$\mathrm{t}^a{}_b = \sum_{k=0}^{\infty}\mathrm{t}^{a(k)}_{b}\lambda^k, \quad \rho = \sum_{k=1}^{\infty}\rho^{(k)}\lambda^k \tag{B.10}$$

As there is only one term of order $\frac{1}{\lambda^2}$ in equation (B.9) it immediately implies that the leading term of $\mathrm{t}^\tau{}_j \sim \mathcal{O}(\lambda)$ and the leading term of $\hat{\mathrm{t}}^i{}_j$ is

$$\hat{\mathrm{t}}^{(1)}_{ij} = 2\mathrm{t}^{\tau(1)}_{i}\mathrm{t}^{\tau(1)}_{j} - 2\mathrm{t}^{\tau(1)}_{(i,j)} - \text{trace}. \tag{B.11}$$

The exact Hamiltonian constraint

$$-2p\partial_a\partial^a\rho + p(1-p)\partial_a\rho\partial^a\rho + e^{-2p\rho}\left[\frac{p+1}{p}(\mathrm{t}^\tau{}_\tau)^2 - \frac{1}{\lambda}e^{(p+1)\rho}\mathrm{t}^\tau{}_\tau - \frac{2}{\lambda^2}\mathrm{t}^\tau{}_i\mathrm{t}^{\tau i} + \hat{\mathrm{t}}^2_{ij}\right] = 0 \tag{B.12}$$

at leading order fixes $\mathrm{t}^\tau{}_\tau$ as

$$\mathrm{t}^{\tau(1)}_{\tau} = -2\mathrm{t}^{\tau(1)}_{i}\mathrm{t}^{\tau i(1)}. \tag{B.13}$$

Finally we come to the momentum constraints

$$\begin{aligned} \partial_a \mathrm{T}^a{}_0 &= \tfrac{1}{\lambda}\partial^i \mathrm{t}^\tau{}_i + \tfrac{p}{2}e^{(p+1)\rho}\partial_\tau\rho - \lambda\partial_\tau \mathrm{t}^\tau{}_\tau = 0, \\ \partial_a \mathrm{T}^a{}_j &= -\partial_\tau \mathrm{t}^\tau{}_j - \partial_i \hat{\mathrm{t}}^i{}_j + \tfrac{1}{p}\partial_j \mathrm{t}^\tau{}_\tau - \tfrac{1}{2\lambda}e^{(p+1)\rho}\partial_j\rho = 0. \end{aligned} \tag{B.14}$$

The time component gives at leading order

$$\partial^i \mathrm{t}^{\tau(1)}_i = 0. \tag{B.15}$$

The space components are at leading order

$$\partial_j \rho^{(1)} = 0 \tag{B.16}$$

and at the next order

$$\partial_\tau \mathrm{t}^{\tau(1)}_i + 2\mathrm{t}^{\tau(1)}_k \partial^k \mathrm{t}^{\tau(1)}_i - \partial^2 \mathrm{t}^{\tau(1)}_i + \frac{1}{2}\partial_j \rho^{(2)} = 0. \tag{B.17}$$

Identifying

$$\mathrm{t}^{\tau(1)}_i = v_i/2, \qquad \rho^{(2)} = P, \tag{B.18}$$

as the velocity and pressure fields, (B.15) and (B.17) become

$$\partial_k v^k = 0, \tag{B.19}$$

$$\partial_\tau v_i + v^k \partial_k v_i - \partial^2 v_i + \partial_i P = 0. \tag{B.20}$$

This again is the incompressible Navier-Stokes system in p space dimensions [64].